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A DERIVATION OF THE LORENTZ FORMULAE

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[Received 19 March 1933]

Introduction

1. In a recent paper Narliker has given a new derivation of the Lorentz formulae of special relativity.* It may therefore not be inopportune to communicate the present note which gives a derivation of the Lorentz formulae from a different starting-point. The considerations underlying the derivation are of a more physical character.

2. In his famous paper 'On the electrodynamics of moving bodies' (1905) Einstein deduced the Lorentz formulae by means of an appeal to the homogeneity of space and time, which amounted to assuming that the transformations in question were linear.† In his idealized experiments he used light signals, clocks, and measuring rods. The idea of a measuring rod involves that of a rigid body, and it was in connexion with such ideas that the actual appeal to homogeneity was made. Minkowski, however, in his equally famous address on 'Space and Time' (1908) drew attention to the fact that while the differential equation for the propagation of light in empty space possesses the group G_c, the concept of rigid bodies has a meaning only for the group G_∞.‡ Consequently proofs of the Lorentz formulae not involving the use of measuring rods or the assumption of linearity have since been sought.

In a highly abstract paper published in 1924 Carathéodory pointed out that the whole investigation can be based on time observations alone provided we use the wave theory of light.§ In his recent paper Narliker considered the transformations for which the wave equation is invariant and discovered that these are all linear.

3. An entirely different line of attack was adopted by Frank and Rothe in 1911.|| They considered the abstract problem of two

* Narliker, *Proc. Camb. Phil. Soc.* 28 (1932), 460–2.

† Einstein, *Ann. der Phys.* 17 (1905), 891–921.

‡ Sommerfeld gives full references to the investigations of Born, Planck, Ehrenfest, and others on this subject in *The Principle of Relativity*, 92, note 1, (Methuen, 1923).

§ Carathéodory, *Sitz. Preuss. Akad.* (1924), 12–27.

|| Frank und Rothe, *Ann. der Phys.* 34 (1911), 825–55.

systems in uniform relative motion determined by the following postulates:

(i) the equations of transformation from one system to another form a linear, homogeneous group, and

(ii) space and time are isotropic in so far as only the magnitude and not the direction of the relative velocity appears in the transformation equations.

As Freundlich has remarked, 'An essential feature is that the constancy of the velocity of light is *not* demanded in either of the postulates (i) or (ii). Rather, the distinguishing property of a *definite* velocity which preserves its value in *all* systems that emerge out of one another through such transformations is a direct corollary to these two general postulates and the result of the Michelson-Morley experiment merely determines the value of this special velocity which could, of course, be found only from observation.'* In 1921, Pars, quite independently, also discovered that the existence of an invariant velocity is a direct consequence of equivalent assumptions.†

In both of these papers the existence of an invariant velocity was deduced from the assumption that the transformations were linear. Strictly speaking, Frank and Rothe began by considering bilinear transformations, which were easily reduced to the more restricted form. The postulate of bilinearity was based on a theorem of Lie's that all transformations which convert points and straight lines in one plane into points and straight lines in another are necessarily homographic. If we consider the transformations which give one-one correspondence between two sets of points on the same line, we see that in this restricted case the theorem is only true when the variables are permitted to assume complex values. In the following treatment of the one-dimensional case this assumption is unnecessary.

The Lorentz formulae for a one-dimensional world

4. *Postulates assumed.* Consider a sub-world of only one degree of spatial freedom.‡ Let it be characterized by the two following postulates:

* Freundlich, *Foundations of Einstein's Theory of Gravitation* 79, (Methuen, 1924).

† Pars, *Phil. Mag.* 42 (1921), 249–58.

‡ No use is made of the notion of 'straightness'. The sub-world considered could be regarded as a twisted curve in the ordinary world, provided certain obvious conventions and modifications were made.

(i) Einstein's postulate: That for all sources and observers in uniform relative motion the velocity of 'light' is invariant.

(ii) Galileo-Newton's postulate: That a family of observers in uniform relative motion possesses no 'privileged' members.*

The first of these postulates will be treated purely as a means of *calculating* distances. The velocity of 'light' will be denoted by an arbitrary constant c and measurements will be calculated on the convention that the distance travelled by a signal is the product of the velocity c and the time of transit.†

Let A and B be any two observers moving in this one-dimensional world. Each has a clock which records zero time at the instant when they are together. A dispatches a blue signal at time t_0 by his clock. On arrival at B it is instantaneously reflected by a mirror attached to B and returns to A at time τ_0 by A's clock. A dispatches a red signal at time $t_0+\delta$ and this returns to him at time $\tau_0+\epsilon$. If the interval of time which elapses between the arrival at B of the two signals is so short that during it the velocity of B with respect to A may be taken as some constant V, A may deduce the relation,

$$c(\epsilon-\delta) = V(\epsilon+\delta),$$

by using postulate (i). By definition, if B is in uniform motion with respect to A, this equation must give the same value to V/c for *all* δ, ϵ. In this manner A can discover if B is in uniform relative motion and can calculate the relative velocity as a fraction of c.

At the instant when A and B are together they agree on the assignation of positive and negative spatial sense, and after they part company B appears to A to be moving positively and, conversely, A appears to B to be moving negatively. Postulate (ii) implies that if the velocity of B according to A is V, then the velocity of A with respect to B is $-V$.‡ Moreover, this postulate implies that the

* This means that the transformations in question form a group. For the precise mathematical significance of this postulate see § 9, especially equations (27) and (28). It is this postulate which contains the essence of the principle of relativity.

† More generally, of course, if the velocity of a body or particle is a constant V, the distance travelled is the product of V and the time of transit. The peculiar significance of c is that it is an *invariant velocity* characterizing the particular world considered; it is not necessarily the velocity of natural light, but it is assumed that signals of some kind may be transmitted with this velocity.

‡ Theoretically this could be used as a method of defining the use by A and B of similar clocks.

transformation-equations connecting the space and time of one system with those of the other are reciprocal, provided the sign of the relative velocity is reversed.

5. *The first experiment.* At an arbitrarily chosen instant t_1 by his clock, A dispatches a signal which *passes* B at τ_1' by B's clock and is ultimately reflected instantaneously by an arbitrary mirror M on the far side of B. (M may be moving in any manner.) On its return journey the signal passes B again at τ_2' according to B and arrives back at A at t_2, by A's clock. If, according to A's calculations, τ_1 is the instant when the outward signal passes B, the following equation (for the distance AB at that instant) is satisfied:

$$c(\tau_1 - t_1) = V\tau_1. \tag{1}$$

Similarly, if τ_2 is the instant, calculated by A, when the return signal passes B, the equation

$$c(t_2 - \tau_2) = V\tau_2 \tag{2}$$

is true. With this information the following time-table can be constructed.

EVENT	B_1	M	B_2
A	$\dfrac{ct_1}{c-V}$	$\frac{1}{2}(t_2+t_1)$	$\dfrac{ct_2}{c+V}$
B	τ_1'	$\frac{1}{2}(\tau_2'+\tau_1')$	τ_2'

TABLE 1

B_1 denotes the first arrival of the signal at B,
M denotes the reflection of the signal at the mirror M,
B_2 denotes the second arrival of the signal at B.

6. If, for any definite event, there were a functional relation between the times t and t', assigned to it by A and B respectively, of the form

$$t' = \phi(t, V), \tag{3}$$

the following functional equation could be deduced from Table 1:

$$\phi\left(\frac{ct_2}{c+V}, V\right) + \phi\left(\frac{ct_1}{c-V}, V\right) = 2\phi\left(\frac{t_2+t_1}{2}, V\right). \tag{4}$$

Since M is arbitrary, it follows that t_1 and t_2 are independent. Assuming that ϕ may be differentiated twice with respect to its first

argument, the operation denoted by $\partial^2/\partial t_1\,\partial t_2$ gives

$$\phi_{tt}(t, V) = 0. \tag{5}$$

Consequently, $t' \equiv \phi(t, V) = a(V)t + b(V).$

Substituting this form in equation (4), we deduce that if c be finite and V not equal to zero, then

$$a(V) = 0.$$

Consequently, t' is independent of t, which is physically absurd. Hence Table 1 does not provide sufficient information for correlating the two clocks. It does suffice, however, to show that

$$t' \neq t, \tag{6}$$

and thus that no physical meaning can be attached to the idea of simultaneity with respect to systems in relative motion, save in the extreme cases of infinite c or zero V.

7. If the idea of distance is introduced (see § 4) the required additional information can be obtained. With the previous notation, A determines the distance from himself of the event M by the equation

$$x = \tfrac{1}{2}c(t_2 - t_1). \tag{7}$$

Similarly, B determines the distance from himself of the same event M by the equation

$$x' = \tfrac{1}{2}c(\tau_2' - \tau_1'). \tag{8}$$

Table 1 can now be replaced by Table 2.

Event	A		B	
	t	x	t'	x'
B_1	$\dfrac{ct_1}{c-V}$	$\dfrac{cVt_1}{c-V}$	τ_1'	0
M	$\tfrac{1}{2}(t_2+t_1)$	$\tfrac{1}{2}c(t_2-t_1)$	$\tfrac{1}{2}(\tau_2'+\tau_1')$	$\tfrac{1}{2}c(\tau_2'-\tau_1')$
B_2	$\dfrac{ct_2}{c+V}$	$\dfrac{cVt_2}{c+V}$	τ_2'	0

Table 2

We now seek whether a consistent description can be obtained by replacing the functional equation (3) by the relation

$$t' = \phi(x, t),$$

where x and t are independent. This suggests seeking a further relation of the form

$$x' = f(x, t). \tag{10}$$

The functions ϕ and f may involve V as well as x and t. From Table 2 the following six functional equations can now be constructed:

$$\begin{cases} \phi[\tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)] = \tfrac{1}{2}(\tau_2'+\tau_1'), & (11) \\ \tau_1' = \phi\left(\dfrac{cVt_1}{c-V}, \dfrac{ct_1}{c-V}\right); \qquad \tau_2' = \phi\left(\dfrac{cVt_2}{c+V}, \dfrac{ct_2}{c+V}\right), & (12) \end{cases}$$

$$\begin{cases} f[\tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)] = \tfrac{1}{2}c(\tau_2'-\tau_1'), & (13) \\ 0 = f\left(\dfrac{cVt_1}{c-V}, \dfrac{ct_1}{c-V}\right); \qquad 0 = f\left(\dfrac{cVt_2}{c+V}, \dfrac{ct_2}{c+V}\right). & (14) \end{cases}$$

Equations (14) may be replaced by the single equation

$$f(Vt, t) = 0, \tag{15}$$

for all t, since t_1 and t_2 are arbitrary.*

If the substitutions

$$\xi = \frac{t_2}{1+\beta} \quad \text{and} \quad \eta = \frac{t_1}{1-\beta}, \tag{16}$$

where

$$\beta = V/c, \tag{17}$$

are made, then equations (11) and (12) lead to the equation

$$\phi(c\beta\xi, \xi)+\phi(c\beta\eta, \eta) = 2\phi(x, t), \tag{18}$$

where

$$x = \tfrac{1}{2}c\{\xi(1+\beta)-\eta(1-\beta)\} \tag{19}$$

and

$$t = \tfrac{1}{2}\{\xi(1+\beta)+\eta(1-\beta)\}. \tag{20}$$

These coordinates refer to *any* event M and so are perfectly general.

8. Operating on equation (18) with the operator $\partial^2/\partial\xi\partial\eta$, we obtain the equation

$$c^2(1-\beta^2)\phi_{xx}(x, t) = (1-\beta^2)\phi_{tt}(x, t). \tag{21}$$

If $V \neq \pm c$, then $\beta^2 \neq 1$, and we may replace equation (21) by the partial differential equation

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \tag{22}$$

This is D'Alembert's wave-equation and its general solution is

$$\phi(x, t) = \Phi(ct+x)+\Psi(ct-x), \tag{23}$$

* This equation merely signifies that all events at B have a zero x'-coordinate. Its use is as a check on our subsequent calculations, e.g. equation (26).

where Φ and Ψ are arbitrary functions. If this form be substituted in equation (18), we obtain the relation

$$\Phi\{c\xi(1+\beta)\}-\Psi\{c\xi(1-\beta)\} = \Phi\{c\eta(1+\beta)\}-\Psi\{c\eta(1-\beta)\} = \text{const.},$$

since ξ and η are independent. Hence, replacing either $c\xi$ or $c\eta$ by any λ and absorbing the additive constant, we obtain the relation

$$\Phi\{(1+\beta)\lambda\} = \Psi\{(1-\beta)\lambda\}.$$

If we define a function Θ by the equations

$$\Theta\{(1-\beta^2)\lambda\} = \Phi\{(1+\beta)\lambda\} = \Psi\{(1-\beta)\lambda\}, \tag{24}$$

the general solution of equation (18) may be written as

$$t' = \phi(x,t,V) = \Theta\{(1-\beta)(ct+x), c\beta\}+\Theta\{(1+\beta)(ct-x), c\beta\}. \tag{25}$$

Using this result in equations (12) and (13) we also obtain

$$x' = f(x,t,V) = c\Theta\{(1-\beta)(ct+x), c\beta\}-c\Theta\{(1+\beta)(ct-x), c\beta\}. \tag{26}$$

Equations (25) and (26) constitute the complete solution of the equations (11) to (14).*

9. The Galileo-Newton postulate implies that if

$$x' = f(x,t,V) \quad \text{and} \quad t' = \phi(x,t,V),$$

then

$$x = f(x',t',-V) \quad \text{and} \quad t = \phi(x',t',-V),$$

i.e.

$$\begin{cases} x = f\{f(x,t,V), \phi(x,t,V), -V\} & (27)\\ t = \phi\{f(x,t,V), \phi(x,t,V), -V\}. & (28)\end{cases}$$

From equations (25) to (28) the relations

$$\begin{cases} x = c\Theta\{(1+\beta)(c\phi+f), -c\beta\}-c\Theta\{(1-\beta)(c\phi-f), -c\beta\} & (29)\\ t = \Theta\{(1+\beta)(c\phi+f), -c\beta\}+\Theta\{(1-\beta)(c\phi-f), -c\beta\}, & (30)\end{cases}$$

where $\phi \equiv \phi(x,t,V)$ and $f \equiv f(x,t,V)$, can be deduced. The equations

$$\begin{cases} c\phi+f = 2c\Theta\{(1-\beta)(ct+x), c\beta\}\\ c\phi-f = 2c\Theta\{(1+\beta)(ct-x), c\beta\}\end{cases}$$

follow from equations (25) and (26). If we substitute

$$\zeta = (1-\beta)(ct+x), \qquad \chi = (1+\beta)(ct-x), \tag{31}$$

* Attention should be drawn to the fact that if Vt is substituted for x in equation (26), then equation (15) is satisfied automatically, showing that the form of Θ is perfectly general, so far, as may also be verified by substituting (25) in (18).

equation (29) may be written in the form

$$c\Theta\{2c(1+\beta)\Theta(\zeta, c\beta), -c\beta\} - \frac{\zeta}{2(1-\beta)}$$

$$= c\Theta\{2c(1-\beta)\Theta(\chi, c\beta), -c\beta\} - \frac{\chi}{2(1+\beta)}. \tag{32}$$

Since x and t are independent it follows that ζ and χ are so as well. Hence each side of equation (32) may be equated to an arbitrary function k *independent* of ζ and χ. Since $x, t = 0$ implies $x', t' = 0$, it follows from equation (25) that

$$\Theta(0, c\beta) = 0. \tag{33}$$

Hence the arbitrary function k must be replaced by zero.

If in the equation which results from equating the right-hand side of equation (32) to zero we replace χ by ζ and β by $-\beta$, we can deduce the relations

$$\Theta\{2c(1+\beta)\Theta(\zeta, c\beta), -c\beta\} = \Theta\{2c(1+\beta)\Theta(\zeta, -c\beta), c\beta\} = \frac{\zeta}{2c(1-\beta)}. \tag{34}$$

Writing $$(1+\beta)/(1-\beta) = \alpha^2, \tag{35}$$

we deduce from (34) the relations

$$\Theta(\alpha^2\zeta, c\beta) = \Theta[2c(1+\beta)\Theta\{2c(1+\beta)\Theta(\zeta, c\beta), -c\beta\}, c\beta] = \alpha^2\Theta(\zeta, c\beta). \tag{36}$$

Omitting explicit reference to $c\beta$, we deduce that Θ must satisfy a functional equation of the form,

$$\psi(\alpha^2\zeta) = \alpha^2\psi(\zeta), \tag{37}$$

where $\alpha^2 \neq 1$, since $\beta \neq 0$.

10. *The functional equation* $\psi(\alpha^2\zeta) = \alpha^2\psi(\zeta)$. Since $\alpha^2 \neq 1$, it is immediately seen that we must have

$$\psi(0) = 0, \tag{38}$$

in accordance with equation (33). Assuming ψ to be differentiable with respect to ζ, we have

$$\psi'(\alpha^2\zeta) = \psi'(\zeta),$$

and more generally $$\psi'(\alpha^{\pm 2r}\zeta) = \psi'(\zeta),$$

for any positive integer r. If ζ_0 be any definite value of ζ, then, whether

$$|\alpha^2| < 1 \quad \text{or} \quad |\alpha^2| > 1,$$

an infinite sequence,

$$\zeta_r = \alpha^{2r}\zeta_0 \quad \text{or} \quad \alpha^{-2r}\zeta_0,$$

converging to zero, can be constructed so that

$$\psi'(\zeta_r) = \psi'(\zeta_0).$$

It follows that if $\psi'(\zeta)$ has a unique limit l as $\zeta \to 0$, this will also be the limit of $\psi'(\zeta_r)$, and hence for *any* ζ_0 we have

$$\psi'(\zeta_0) = l. \tag{39}$$

Consequently ψ satisfies an ordinary differential equation, which on integration with the condition (38) gives*

$$\Theta(\zeta, c\beta) \equiv \psi(\zeta) = l(V)\zeta. \tag{40}$$

Substituting this result in equation (34), we obtain the condition

$$l(V)l(-V) = \frac{1}{4(c^2-V^2)}. \tag{41}$$

Substituting the form of Θ given by equation (40) in equations (25) and (26), we have the formulae

$$\left.\begin{aligned} x' &\equiv f(x,t,V) = 2cl(V)(x-Vt) \\ t' &\equiv \phi(x,t,V) = 2cl(V)(t-Vx/c^2) \end{aligned}\right\}. \tag{42}$$

Conversely, we have

$$\left.\begin{aligned} x &\equiv f(x',t',-V) = 2cl(-V)(x'+Vt') \\ t &\equiv \phi(x',t',-V) = 2cl(-V)(t'+Vx'/c^2) \end{aligned}\right\}. \tag{43}$$

Putting $x = 0$ in the second of equations (42), we get

$$t' = 2cl(V)t. \tag{44}$$

Hence, if B is moving away from A with uniform relative velocity V, equation (44) gives B's calculation of the time of any event at A. Since A is moving away from B with the same relative velocity, the reciprocal equation must be

$$t = 2cl(V)t'.$$

But by putting $x' = 0$ in the second of equations (43), the reciprocal equation is seen to be

$$t = 2cl(-V)t'.$$

Consequently, $$l(V) = l(-V),$$

* It can be shown that $\alpha^2 \neq 1$ and the existence of $\lim \psi'(\zeta)$ as $\zeta \to 0$ are necessary as well as sufficient conditions for (40) to be the complete solution of (37). Since we have already assumed that ϕ and f may be differentiated twice with respect to x and t, the limit condition is not a new assumption.

and hence, by equation (41),

$$l(V) = \pm\tfrac{1}{2}(c^2-V^2)^{-\frac{1}{2}}.$$

The x'-coordinate of any event at A is given by

$$x' = -2cl(V)Vt.$$

From the conventions made with respect to spatial sense this must be of the opposite sign to Vt. Consequently,

$$l(V) = +\tfrac{1}{2}(c^2-V^2)^{-\frac{1}{2}}, \tag{45}$$

and we obtain the transformation formulae

$$\begin{cases} x' = (x-Vt)(1-V^2/c^2)^{-\frac{1}{2}}, & (46) \\ t' = (t-Vx/c^2)(1-V^2/c^2)^{-\frac{1}{2}}, & (47) \end{cases}$$

uniquely. These, of course, are the restricted Lorentz formulae.

The three-dimensional problem

11. *The second experiment.* As before, A and B are two observers in uniform relative motion in the line AB. C is any third observer at rest with respect to A, collinear with A and B and between them. A's clock and C's keep pace with each other (see § 6), and this suggests the following experiment. B dispatches a signal at any instant in the direction BA. It arrives at C at time t_1 by C's (and A's) clock and is instantaneously reflected in any direction orthogonal to AB. It is ultimately reflected back to C by an arbitrary mirror N, which may be moving in any manner. The signal returns to C at time t_2 by A's clock and is then reflected back to B. As before, A's clock and B's are synchronized so as to read zero time at the instant when A and B part company. B calculates the instants of the arrival at C of the signal on its outward and inward journeys as t_1', t_2' respectively.

12. The only essentially new calculation connected with this experiment is B's determination of CN. If u is B's calculation of the component of the signal's velocity in the direction CN when travelling from C to N, then, since $-V$ is its component orthogonal to CN, we have, by Einstein's postulate,*

$$u^2+V^2 = c^2,$$

whence

$$u = c(1-\beta^2)^{\frac{1}{2}}. \tag{48}$$

Consequently B's calculation of the distance CN is given by

$$y' = \tfrac{1}{2}c(t_2'-t_1')(1-\beta^2)^{\frac{1}{2}}. \tag{49}$$

* We now assume that the world considered is Euclidean.

The following table can now be constructed.

EVENT	A			B		
	t	x	y	t'	x'	y'
C_1	t_1	x_0	0	t'_1	$x'_0-Vt'_1$	0
N	$\frac{1}{2}(t_2+t_1)$	x_0	$\frac{1}{2}c(t_2-t_1)$	$\frac{1}{2}(t'_2+t'_1)$	$x'_0-\frac{1}{2}V(t'_2+t'_1)$	$\frac{1}{2}c(t'_2-t'_1)(1-\beta^2)^{\frac{1}{2}}$
C_2	t_2	x_0	0	t'_2	$x'_0-Vt'_2$	0

TABLE 3

C_1 denotes the first arrival of the signal at C,
N denotes the reflection of the signal at N,
C_2 denotes the second arrival of the signal at C,
x_0 is A's reckoning of the distance AC, and x'_0 is B's.

13. Considering the plane $NACB$ and taking the x-axis in the direction AB and the y-axis parallel to NC, the relations

$$x' = f(x,y,t), \qquad y' = g(x,y,t), \qquad t' = \phi(x,y,t), \tag{50}$$

when applied to Table 3, give nine functional equations of which the three involving ϕ are:

$$t'_2+t'_1 = 2\phi\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\}, \tag{51}$$

$$t'_1 = \phi(x_0, 0, t_1), \text{ and } t'_2 = \phi(x_0, 0, t_2). \tag{52}$$

It must be emphasized that though x_0 is a constant it is an *arbitrary* constant.

Equations (51) and (52) give the functional equation

$$\phi(x_0, 0, t_1)+\phi(x_0, 0, t_2) = 2\phi\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\}. \tag{53}$$

Since, by equation (47),

$$\phi(x_0, 0, t) = (t-Vx_0/c^2)(1-V^2/c^2)^{-\frac{1}{2}},$$

equation (53) may be replaced by

$$\{\tfrac{1}{2}(t_2+t_1)-Vx_0/c^2\}(1-V^2/c^2)^{-\frac{1}{2}} = \phi\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\}.$$

Since x_0, $\frac{1}{2}c(t_2-t_1)$, and $\frac{1}{2}(t_2+t_1)$ are independent and arbitrary, it follows that ϕ is independent of its second argument, whence

$$t' \equiv \phi(x,y,t) = \phi(x,0,t) = (t-Vx/c^2)(1-V^2/c^2)^{-\frac{1}{2}}. \tag{54}$$

Substituting for t'_2 and t'_1 in the equation

$$g\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\} = \tfrac{1}{2}c(t'_2-t'_1)(1-V^2/c^2)^{\frac{1}{2}}, \tag{55}$$

which is deduced from the middle row of Table 3, it is seen that

$$g\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\} = \tfrac{1}{2}c(t_2-t_1),$$

and hence that

$$g(x,y,t) = y. \tag{56}$$

Similarly,

$$2f\{x_0, \tfrac{1}{2}c(t_2-t_1), \tfrac{1}{2}(t_2+t_1)\} = f(x_0, 0, t_1)+f(x_0, 0, t_2). \tag{57}$$

Using equation (46) we are led to the result

$$x' \equiv f(x,y,t) = f(x,0,t) = (x-Vt)(1-V^2/c^2)^{-\frac{1}{2}}. \tag{58}$$

Equation (56) signifies that A and B agree on all spatial calculations orthogonal to AB. With this information the general Lorentz formulae

$$\left.\begin{aligned} x' &= (x-Vt)(1-V^2/c^2)^{-\frac{1}{2}} \\ y' &= y, \qquad z' = z \\ t' &= (t-Vx/c^2)(1-V^2/c^2)^{-\frac{1}{2}} \end{aligned}\right\} \tag{59}$$

and may be immediately deduced.

Summary

14. We have considered a world characterized by the following postulates:

(i) The existence of a definite velocity c which is invariant for all observers A, B,... in uniform relative motion.

(ii) The existence of physical entities ('signals') which may be transmitted with this velocity.

(iii) The Galileo-Newton postulate of relativity.

We have assumed that the transformation formulae may be differentiated twice. Moreover, we have assumed that space and time are isotropic in so far as the orientation of AB is irrelevant. This assumption is implicit in condition (iii), for if it were not true then there would exist preferential directions, contrary to the spirit of relativity. With these conditions and assumptions we have proved that the transformation formulae are of the Lorentz form.* This we have shown by taking clocks as fundamental, but it would have been *logically* equivalent to have taken taut measuring tapes folding back to the observer who halves the measurements he reads off.

In conclusion I should like to thank Professor Milne for his kindness in giving help and advice. The idealized experiment of § 5 and the problems arising from it were originally suggested by him.

* If we make the additional assumption that there are no universal constants of time or distance the work from § 9 onwards may be replaced by a simpler dimensional argument. Without this assumption our work is equivalent to the proof that no such constants exist.

ON AN INTERPOLATED INTEGRAL FUNCTION OF GIVEN ORDER

By M. MURSI (*Cairo*) and C. E. WINN (*Cairo*)

[Received 14 December 1932]

It is well known* that, given a real increasing sequence r_n whose only limit is infinity, together with an associated complex sequence b_n, there exists an integral function $f(z)$ such that

$$f(r_n) = b_n \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

We observe at the outset that, if the modulus of b_n increases on the whole so rapidly that

$$\overline{\lim_{n\to\infty}} \frac{\log\log \mu(r)}{\log r} = \sigma > 0, \tag{2}$$

where $$\mu(r) = \max_{1 \leqslant \nu \leqslant n} |b_\nu| \qquad (r_n \leqslant r < r_{n+1}),$$

then the order of $f(z)$ cannot be less than σ, since for certain values of n, as great as we please,

$$f(r_n) = b_n = \mu(r_n).$$

This sets a natural lower limit on the order of the constructed function. On the other hand the order of such a function is also affected by the proximity of the numbers r_n. For example, when

$$b_{2n-1} = 0, \qquad b_{2n} = 1 \qquad (n = 1, 2, 3, \ldots),$$

whilst $$r_n = n,$$

we can construct a function of unit order to satisfy (1). But if, with the same values of b_n, and

$$r_{2n-1} = n, \qquad r_{2n} = n + e^{-n^c},$$

where $c > 1$, the order of the derivate and therefore of the function itself would be greater than unity.

These remarks suffice to show the necessary dependence of the order of $f(z)$ on the distribution of the r_n's. The function $f(z)$ is the canonical product with simple zeros r_n multiplied by a meromorphic function having only simple poles at r_n with residue equal to b_n divided by the derivate of the product at r_n. In general, unless the b_n's happen to be of a very special character,† the order of the

* See Osgood, *Functiontheorie*, p. 540.

† For example, when $b_n = \exp(n^{1/\rho})$, $r_n = n^{1/\rho} (\rho > 1)$, the function $f(z) = e^z$ takes the values b_n at r_n, although its order is less than ρ.

function cannot be less than ρ, the exponent of convergence of r_n, so that the order of $f(z)$ is not less than $\max(\rho, \sigma)$. If either ρ or σ is infinite, the order of $f(z)$, being infinite, is of course equal to $\max(\rho, \sigma)$. The question was suggested to one of us by Dr. J. M. Whittaker as to whether the same could be said, when both ρ and σ are finite. And it is our object here to show that for certain sequences r_n there are integral functions with the property (1), whose order is equal to the greater of ρ and σ.

We consider the case for which the increase of r_n is restricted by the relation*

$$\lim_{n\to\infty} \frac{\log n}{\log r_n} = \rho > 0, \tag{3}$$

which of course implies

$$\lim_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} = 1. \tag{4}$$

Further, we limit the proximity of the values r_n by the condition

$$\underline{\lim}_{n\to\infty}\, n\frac{r_{n+1}-r_n}{r_n} > 0. \tag{5}$$

The first step in our work is to construct an integral function $J(z)$ of order σ such that

$$J(r_n) > |b_n|. \tag{6}$$

We then form an integral function $\phi(z)$ of order ρ, with simple zeros r_n, such that

$$n|\phi'(r_n)| > \tfrac{1}{2}. \tag{7}$$

The required integral function is then given by

$$\chi(z) = J(z)\phi(z)\sum_{n=1}^{\infty} \frac{b_n}{(z-r_n)\phi'(r_n)J(r_n)}\left(\frac{z}{r_n}\right)^q,$$

where q is an integer chosen great enough to secure the convergence of the series.

We may suppose without loss of generality that $r_1 > 1$. For otherwise the finite number of r's not exceeding unity can be allowed for by a polynomial factor incorporated in $\phi(z)$, which will not affect the order of the constructed function. Accordingly,

$$\rho(r_n) = \frac{\log n}{\log r_n} \tag{8}$$

may be assumed positive. Let us consider the polygonal function

* In general, the exponent ρ is equal to $\overline{\lim}_{n\to\infty} \dfrac{\log n}{\log r_n}$. The values of r_n here considered increase like the zeros of a function of regular growth.

$\rho(r)$ which joins the points $\{r_n, \rho(r_n)\}$, so that $\rho(r)$ is continuous, and, in view of (3), tends to ρ, as $r \to \infty$. We then derive a monotonic increasing function*

$$\lambda(x) = \min_{r \geqslant x} \rho(r)$$

whose limit is also ρ. Moreover, if h is any positive number less than ρ/σ, we can fix r' so that for $r \geqslant r'$,

$$\frac{\lambda(x)}{\sigma} > h' > h. \tag{9}$$

Next, putting
$$\frac{\log\log \mu(r_n)}{\log r_n} = \sigma(r_n),$$

we form the polygonal function $\sigma(r)$ through the points $\{r_n, \sigma(r_n)\}$, and derive the monotonic decreasing function*

$$\tau(x) = \max_{r \geqslant x^h} \sigma(r),$$

which by (2) tends to σ, as $r \to \infty$.

Now consider the equation

$$x = e^{-1}(Ar_n)^{\tau(x)/\lambda(x)}, \tag{10}$$

where A is a constant at our disposal, and r_n is taken great enough to yield a root $x \geqslant r'$. We may suppose

$$r_N \leqslant x < r_{N+1}, \tag{11}$$

so that N tends to infinity with n. Thus we have

$$\log r_N + 1 \leqslant \{\tau(r_N)/\lambda(r_N)\}\log(Ar_n). \tag{12}$$

Let us take the integral function

$$J(z) = \sum_{n=1}^{\infty} \{Ar_n^{-\lambda(r_n)/\tau(r_n)} z\}^n.$$

Its order is seen to be σ. For† on account of (3) and the limiting values of $\lambda(r_n)$ and $\tau(r_n)$,

$$\varlimsup_{n\to\infty} \frac{n\{\lambda(r_n)/\tau(r_n)\}\log r_n - n\log A}{n \log n} = \frac{1}{\sigma}.$$

We next establish (6). Since the expansion of $J(z)$ consists of positive terms only, we have

$$J(r_n) > (Ar_n)^N \{r_N^{-\lambda(r_N)/\tau(r_N)}\}^N,$$

* By max and min are understood the upper and lower bounds respectively, when there is no greatest or least value.

† G. Valiron, *Lectures on the General Theory of Integral Functions* (1923), pp. 40–1; cf. also pp. 38–9.

whence, using (12), we get

$$\log J(r_n) > N \log Ar_n - N\frac{\lambda(r_N)}{\tau(r_N)}\log r_N$$

$$\geqslant N\frac{\lambda(r_N)}{\tau(r_N)} > \frac{N+1}{k}$$

$$= k^{-1} r_{N+1}^{\rho(r_{N+1})},$$

where k is a positive constant. Further, by (11) and (10), the last expression exceeds

$$k^{-1}e^{-\rho(r_{N+1})}(Ar_n)^{\frac{\tau(x)}{\lambda(x)}\rho(r_{N+1})}. \tag{13}$$

But $$\rho(r_{N+1}) \geqslant \min_{r \geqslant x} \rho(r) = \lambda(x),$$

and $$\tau(x) \geqslant \tau(r_{N+1}) = \max_{r \geqslant r_{N+1}^h} \sigma(r),$$

which is $\geqslant \sigma(r_n)$, provided that $r_n \geqslant r_{N+1}^h$. Now for sufficiently large values of n (and therefore of N) it follows from (10) and (9) that

$$r_n = A^{-1}(ex)^{\lambda(x)/\tau(x)} > A^{-1}(er_N)^{h'},$$

which in virtue of (4) will exceed r_{N+1}^h, when n is great enough. Thus the exponent of Ar_n in (13) eventually exceeds $\sigma(r_n)$, and therefore, by suitable choice of A, we can obtain for all values of n

$$\log J(r_n) > r_n^{\sigma(r_n)}$$

$$= \log \mu(r_n) \geqslant \log|b_n|,$$

which gives (6).

We next construct $\phi(z)$. The condition (5) shows that for some positive δ and all $n \geqslant 1$

$$\frac{r_{n+1}-r_n}{r_n} > \frac{\delta}{n}. \tag{14}$$

We take an integer $p > \rho$ and $3/\delta$. Then with $r_n = a_n^{(0)}$ we associate the complex numbers

$$a_n^{(q)} = r_n e^{(2q/p)\pi i} \qquad (1 \leqslant q \leqslant p).$$

The required function is

$$\phi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n^{(0)}}\right)\left(1-\frac{z}{a_n^{(1)}}\right)\cdots\left(1-\frac{z}{a_n^{(p)}}\right)$$

$$= \prod_{n=1}^{\infty}\left(1-\frac{z^p}{r_n^p}\right) = \prod_{n=1}^{\infty}\left(1-\frac{Z}{R_n}\right),$$

where $Z = z^p$ and $R_n = r_n^p$. The convergence of the product is ensured by the condition $p > \rho$, whilst the order of $\phi(z)$ is equal to

the exponent of r_n, namely ρ. Further, we have

$$\phi'(r_n) = \lim_{z\to r_n}\frac{\phi(z)}{z-r_n} = p\prod_{\nu=1}^{\infty}{}'\left(1-\frac{R_n}{R_\nu}\right),$$

the value $\nu = n$ being excluded. For

$$\prod_{q=1}^{p-1}\left(1-\frac{r_n}{a_n^{(q)}}\right) = \prod_{q=1}^{p-1}(1-e^{-(2q/p)\pi i}) = \lim_{x\to 1}\frac{1-x^p}{1-x} = p.$$

Also, from (14) and our choice of p we deduce that

$$\frac{R_{n+1}-R_n}{R_n} = \frac{r_{n+1}^p-r_n^p}{r_n^p} > \frac{p(r_{n+1}-r_n)}{r_n} > \frac{3}{n}.$$

Hence
$$\frac{R_{n+1}}{R_n} > 1+\frac{3}{n} > \left(\frac{n+1}{n}\right)^2,$$

so that for $n \neq \nu$ we get

$$\left|1-\frac{R_n}{R_\nu}\right| > \left|1-\frac{n^2}{\nu^2}\right|,$$

$$\prod_{\nu=1}^{\infty}{}'\left|1-\frac{R_n}{R_\nu}\right| > \prod_{\nu=1}^{\infty}{}'\left|1-\frac{n^2}{\nu^2}\right|.$$

By considering the limiting value of the function* $(\sin\pi z)/\pi z$, as $z \to n$, we find the last product to be equal to $1/2n$, whence we arrive at (7).

We now proceed to show that the constructed function $\chi(z)$ satisfies the conditions of the problem. We first prove that the series

$$\sum_{n=1}^{\infty}\frac{\phi(z)b_n}{(z-r_n)\phi'(r_n)J(r_n)}\left(\frac{z}{r_n}\right)^q, \tag{15}$$

where q is a positive integer $> 3\rho$, is absolutely and uniformly convergent in the finite part of the z-plane, and that its sum is an integral function of order not exceeding ρ.

We first observe that, for $|z-r_n| > 1$,

$$\left|\frac{\phi(z)}{z-r_n}\right| \leqslant |\phi(z)| \leqslant M(r),$$

* It is of interest to note that in the special case $r_n = n$, it is sufficient in the above to take $p = 2$, yielding

$$\phi(z) = (\sin\pi z)/\pi z.$$

the maximum of $|\phi(z)|$, when $|z| = r$. On the other hand, for $|z-r_n| \leqslant 1$, the modulus of

$$\frac{\phi(z)}{z-r_n}$$

assumes its maximum value on the boundary. In either case, therefore, we have

$$\left|\frac{\phi(z)}{z-r_n}\right| \leqslant \max_{|z|=r+2} |\phi(z)|$$
$$= M(r+2),$$

which is clearly of order ρ. In view, then, of (6) and (7), (15) has as dominant the series

$$\frac{r^q}{\delta} M(r+2) \sum_{n=0}^{\infty} \frac{n}{r_n^q};$$

and the series converges for the chosen value of q on account of (3), with a sum of order ρ.

We see thus that the order of $\chi(z)$ is less than or equal to $\max(\rho, \sigma)$. Also, as $\chi(r_n) = b_n$, it follows from (2) that, for any positive ϵ,

$$\log|\chi(z)| > |z|^{\sigma-\epsilon}$$

for real values of z, arbitrarily large. Consequently, if $\sigma \geqslant \rho$, the order of $\chi(z)$ is equal to $\max(\rho, \sigma)$.

Again in the direction $\quad \arg z = 0$

it is clear that $$\phi(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^p}{r_n^p}\right)$$

assumes its maximum for given $|z|$. Hence, for any $\epsilon > 0$, we can obtain positive values of z, as large as we please, such that

$$\phi(z) > \exp(|z|^{\rho-\epsilon}).$$

Also $J(z)$ is positive when z is positive. Accordingly, when $\sigma < \rho$, the order of $\chi(z)$ is equal to that of $\phi(z)$, namely ρ. The conditions of our problem have thus been fulfilled.

[We are indebted to M. Hadamard for several criticisms, and in particular for pointing out that one case of this problem was treated by Borel.* The latter established the existence of a (unique) interpolated function of order σ, when the growth of the b_n's is sufficiently slow—roughly speaking, when σ is smaller than ρ.]

* *Comptes Rendus*, 124 (1897), 673. Borel's interpolated function is made determinate by the condition of slowest growth.

ON THE SPHERICALLY SYMMETRIC FIELD IN RELATIVITY. II

By B. HOFFMANN (*Rochester, N.Y.*)

[Received 4 November 1932]

1. Introduction

IN a previous paper* by the author it was shown that the most general spherically symmetric field of gravitation and electromagnetism outside matter must, according to the theory of relativity, be a static field so that no energy can be dissipated in the form of spherically symmetric waves *in vacuo*. The general solution of the field equations was obtained under the limitations imposed.

However, in that paper no mention was made of the cosmological constant, and it is the purpose of this note to show how the generalization to the case in which a cosmological term is present in the field equations can be made. It turns out that the generalized problem can be solved along the lines employed in the solution of the previous one, and so it will suffice if we here merely outline the steps of the solution where these steps are essentially the same as those of the earlier paper; equations will be numbered in the present paper by the number of the analogous equation in the other paper, but with an accent attached whenever an alteration has been made.

2. The field equations

The field equations that are to be used here are

$$G_{ab}+E_{ab}+\lambda g_{ab} = 0 \qquad \Phi'(ab)$$

and

$$\frac{\partial(F^{ab}\sqrt{-g})}{\partial x^b} = 0, \qquad \Phi(a)$$

where λ is a constant and the other symbols have their previous significance.

The amended field equations may still be considered as belonging to the unified field theories previously cited† although the cosmological term does not seem to have been specifically included in these theories; the addition of a cosmological constant would, of course,

* B. Hoffmann, *Quart. J. of Math.* (Oxford), 3 (1932), 226–237.

† See also the theory of Schouten and van Danzig developed in a series of papers in *Zeitschrift für Physik*, 1932–3.

not invalidate the theories, and it would leave us with the equations $\Phi'(ab)$ and $\Phi(a)$ that we propose to use as our basis.

3. Spherical symmetry

The question of spherical symmetry is not affected by the inclusion of the cosmological term; the argument of § 3 of the previous paper may be followed through here with the substitution of $(G_{ab}+\lambda g_{ab})$ where we had G_{ab}, since if g_{ab} is spherically symmetric not only is G_{ab} but so also is $(G_{ab}+\lambda g_{ab})$.

In this way we arrive at the result that E_{ab} must satisfy conditions like (8).

4. Proof that the gravitational part of the field is static

The argument of § 4 of the previous paper may be taken over so far as showing that if E_{ab} is spherically symmetric the only possible non-zero components of F_{ab} are $F_{14} = -F_{41}$ and $F_{23} = -F_{32}$; we then obtain as before the results that

$$E_{14} = 0, \qquad E_1^1 = E_4^4.$$

From $\Phi'(14)$, $\Phi'(11)$, and $\Phi'(44)$ we now find

$$(G_{14}+\lambda g_{14}) = 0, \qquad (G_1^1+\lambda g_1^1) = (G_4^4+\lambda g_4^4).$$

But inasmuch as $g_1^1 = \delta_1^1 = 1 = \delta_4^4 = g_4^4$ and we have taken a coordinate system in which $g_{14} = 0$, these equations give us

$$G_{14} = 0, \qquad G_1^1 = G_4^4,$$

so that, as before, the line element is reducible to the static form.

We have thus shown that a spherically symmetric field of gravitation and electromagnetism in the relativity theory, when the cosmological term is not taken to be zero, must be such that the gravitational part is static.

5. The most general spherically symmetric field outside matter

Changing the coordinates by means of a transformation of the form (16), we obtain the line element in the form

$$ds^2 = A\,dt^2 - B\,dr^2 - r^2(d\theta^2+\sin^2\theta\,d\phi^2), \tag{17}$$

where A and B are functions of r alone.

With this coordinate system the conditions of spherical symmetry are still valid and lead to the same result, namely, that $F_{14} = -F_{41}$

and $F_{23} = -F_{32}$ are the only non-zero components of F_{ab}. The equations $\Phi(a)$ and $\Phi(abc)$ are now immediately integrable and give

$$F_{14} = \frac{\epsilon}{r^2}\sqrt{AB}, \qquad F_{23} = \mu \sin\theta,$$

where ϵ and μ are constants of integration. The values of E_{ab} are again given by (18) and (19).

We must now attack the field equations directly; the computation is slightly facilitated if we rewrite these equations in an equivalent form; for, multiplying by g^{ab} and contracting with respect to a and b, we find, since $g^{ab}E_{ab} \equiv 0$ and $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$, that

$$R = 4\lambda,$$

so that the field equations $\Phi'(ab)$ may be replaced by the set

$$R_{ab} - \lambda g_{ab} + E_{ab} = 0; \qquad \Phi''(ab)$$

it is found, on computing the components of R_{ab}, that the only surviving field equations of the set $\Phi''(ab)$ for the new static coordinate system and metric of (17) are, using accents to denote derivation with respect to r,

$$\frac{1}{2}\frac{A''}{A} - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{1}{4}\frac{A'B'}{AB} - \frac{B'}{rB} + \lambda B - \frac{1}{2}B\frac{\epsilon^2+\mu^2}{r^4} = 0, \qquad \Phi''(11)$$

$$\frac{1}{B}\left\{1 + \frac{1}{2}r\left(\frac{A'}{A} - \frac{B'}{B}\right)\right\} - 1 + \lambda r^2 + \frac{1}{2}\frac{\epsilon^2+\mu^2}{r^2} = 0, \qquad \Phi''(22)$$

and

$$\frac{A}{B}\left\{-\frac{1}{2}\frac{A''}{A} + \frac{1}{4}\left(\frac{A'}{A}\right)^2 + \frac{1}{4}\frac{A'B'}{AB} - \frac{A'}{rA}\right\} - \lambda A + \frac{1}{2}A\frac{\epsilon^2+\mu^2}{r^4} = 0, \quad \Phi''(44)$$

the equation $\Phi''(33)$ giving equation $\Phi''(22)$ over again.

From $\Phi''(11)$ and $\Phi''(44)$ we have at once that

$$\frac{1}{r}\left(\frac{B'}{B} + \frac{A'}{A}\right) = 0,$$

i.e. that
$$AB = \text{const.}$$

We may take the constant to be unity by a suitable alteration in the scale of r or t, and thus we have

$$B = A^{-1}.$$

Substituting in $\Phi''(22)$ we find

$$A+rA'-1+\lambda r^2+\frac{1}{2}\frac{\epsilon^2+\mu^2}{r^2}=0,$$

which, on integration, gives

$$Ar-r+\frac{1}{3}\lambda r^3-\frac{1}{2}\frac{\epsilon^2+\mu^2}{r}=\text{const.}=-2m,$$

or

$$\left.\begin{aligned} A &= 1-\frac{2m}{r}-\frac{1}{3}\lambda r^2+\frac{1}{2}\frac{\epsilon^2+\mu^2}{r^2}, \\ &\text{and therefore also} \\ B &= \left\{1-\frac{2m}{r}-\frac{1}{3}\lambda r^2+\frac{1}{2}\frac{\epsilon^2+\mu^2}{r^2}\right\}^{-1} \end{aligned}\right\}.$$

On substituting these values for A and B in $\Phi''(11)$ it is found that the equation is satisfied. Hence the most general spherically symmetric field of gravitation and electromagnetism in the absence of matter, the cosmological term being taken into account, is reducible to the form

$$\left.\begin{aligned} & ds^2 = A\,dt^2-A^{-1}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2), \\ \text{with}\quad & F_{14}=\epsilon r^{-2} \quad\text{and}\quad F_{23}=\mu\sin\theta, \\ \text{where}\quad & A=1-\frac{2m}{r}-\frac{1}{3}\lambda r^2+\frac{1}{2}\frac{\epsilon^2+\mu^2}{r^2}. \end{aligned}\right\} \qquad (20')$$

6. Interpretation of the field

The field we have just obtained corresponds, as before, to the effect of a charged, spherically symmetric piece of matter, not necessarily stationary, having a magnetic pole strength, the mass being measured by m, the electric pole strength by ϵ, and the magnetic pole strength by μ. If isolated magnetic poles be considered as having no existence for the experimental physicist we must set $\mu = 0$ in (20′).

7. Conclusion

In the previous paper Birkhoff's theorem was generalized to refer to the case of a field of gravitation and electromagnetism; in the present note it has been shown that the theorem is also valid for such a field in a universe having non-zero curvature, and the most general line element obeying the restrictions imposed has been obtained. Thus for this more general case it is still true that spherically symmetric waves of gravitation and electromagnetism cannot exist

and that therefore an isolated spherically symmetric distribution of matter cannot lose energy in the form of such radiation.

Correction.

In the previous paper the discovery of the field of a charged sphere was ascribed* to H. Weyl. It has been pointed out that the credit for being the first person to obtain this field is due to H. Reissner,† and the writer wishes to tender his apologies to Professor Reissner for the error—an error that permeates much of the English literature of the relativity theory.

* Hoffmann, ibid., p. 235.

† *Ann. der Physik*, 50 (1916), 106.

ON THE LIMITS OF VALIDITY OF A THEOREM OF STOKES REGARDING THE FIGURE OF THE EARTH

By CORRADINO MINEO (*Palermo*)

[Received 22 January 1933]

THE object of this paper is to examine how far a theorem of Stokes concerning the determination of the figure of the Earth is valid. In a remarkable Memoir, Poincaré attempted to enlarge its validity.† But unhappily the analysis of the great mathematician requires certain further developments, and it will be proved that, on account of the omission of terms, the approximation remains fundamentally the same as Stokes's.‡ Nevertheless, it is possible to remove the restriction that the angular velocity of the planet must be a small quantity of the first order. Even for non-slow rotations (compatible with the dynamical equilibrium of the surface), the problem is soluble and reducible to the third boundary value problem of the potential theory.

1. Suppose that the Earth's gravity has been determined over the whole of an *external* surface of equilibrium S, geometrically unknown but accessible. By means of these values of gravity (and of any other data) is it possible to deduce the equation of S?

Let the Earth be referred to polar coordinates r, θ, ϕ, the origin O coinciding with its centre of mass and situated in the axis of rotation. We may conceive, for instance, that gravity has been determined on S in terms of the angles θ and ϕ. If

$$r = F(\theta, \phi)$$

is the equation of S, the function which is required is $F(\theta, \phi)$.

The problem of determining $F(\theta, \phi)$ may be reduced to the determination of the Newtonian potential V of the Earth outside the surface S. This potential V has to satisfy the following conditions:

(*a*) it must take preassigned values on the *unknown* surface S;

(*b*) it must have a preassigned normal derivative on S.

† See Poincaré, 'Les mesures de gravité et la géodésie': *Bulletin astronomique*, 18 (1901).

‡ Poincaré's analysis is given by Hopfner in his treatise *Physikalische Geodäsie* (Leipzig, 1933) 411–17.

If (a) alone be required, the function F (viz. the surface S) remains wholly arbitrary, and the problem of determining V is then an external Dirichlet's problem.

If (b) alone be required, the surface S remains wholly arbitrary, and the determination of V is then reduced to an external Neumann's problem.

If (a) and (b) are both required, the surface S cannot be arbitrary, and its determination constitutes a kind of *converse* Neumann-Dirichlet problem, for the boundary values are given, but the boundary itself is unknown. The function F must verify a complicated integro-differential equation, which I considered a few years ago. In general, it is not known whether such an equation admits of solutions.†

2. A remarkable particular solution, as is well known, was first given by Stokes. Suppose firstly that the equation of S is

$$r = a(1-\alpha t), \tag{1}$$

where t is a given function of θ and ϕ, and α is a small positive quantity (less than 1). The mass M of the Earth and its angular velocity ω being known, the value of gravity g is uniquely determined on S, according to the hypothesis (1). Let, instead, the observed gravity be g^*, and suppose

$$g^*-g = \beta G(\theta,\phi), \tag{2}$$

where $G(\theta,\phi)$ is a given function and β a small positive constant. The existence of the anomaly g^*-g proves that the hypothesis (1) is not exact. For a second approximation, we suppose that the equation of an external surface of equilibrium S^* of the Earth is

$$r = a(1-\alpha t-\beta u), \tag{3}$$

where u is a function of θ and ϕ, to be determined. *If the terms of the order $\beta\alpha$, $\beta\omega^2$, β^2 are rejected, and if the given function G verifies a condition (which may be expressed by saying that no term of the order* 1 *figures in its expansion in series of Laplace's functions), it follows that there is a unique function, that is to say, a unique surface S^*, to which corresponds the gravity g^*, the total mass and the centre of mass of the Earth remaining unchanged.* This is the theorem of Stokes.‡

† See Mineo, 'Sulla formola di Stokes che serve a determinare la forma della Terra': *Rendiconti del Circolo Matematico di Palermo*, 51 (1927).

‡ See Stokes, *On the Variation of Gravity at the Surface of the Earth* (Cambridge University Press, 1883) 139–42.

3. Let us pass to Poincaré's analysis. After repeating the analysis of Stokes, Poincaré takes as reference surface an ellipsoid of three unequal axes and sets out to reject only terms of the order of the square of the distance between geoid and ellipsoid (loc. cit., pp. 33–9). In this analysis, Lamé's functions play the role which Laplace's do when the reference surface is a sphere.

Let
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{4}$$
be the equation of the ellipsoid S of reference, turning round the shortest axis with constant angular velocity ω. Putting
$$\rho_0^2-a^2=\alpha^2, \qquad \rho_0^2-b^2=\beta^2, \qquad \rho_0^2-c^2=\gamma^2, \tag{5}$$
we consider the family of confocal quadrics
$$\frac{x^2}{\lambda^2-\alpha^2}+\frac{y^2}{\lambda^2-\beta^2}+\frac{z^2}{\lambda^2-\gamma^2}=1, \tag{6}$$
and we call ρ, μ, ν the confocal coordinates of the point (x,y,z), where
$$+\infty>\rho^2>\gamma^2>\mu^2>\beta^2>\nu^2>\alpha^2>0. \tag{7}$$
For $\lambda=\rho=\rho_0$, we have the ellipsoid (4).

Let us put
$$H=\frac{d\rho}{dn}=\frac{\sqrt{(\rho^2-\alpha^2)(\rho^2-\beta^2)(\rho^2-\gamma^2)}}{\rho\sqrt{(\rho^2-\mu^2)(\rho^2-\nu^2)}}. \tag{8}$$
If $F(\rho,\mu,\nu)$ is an arbitrary function, its normal derivative on the ellipsoid of parameter ρ is given, as is well known, by
$$\frac{dF}{dn}=\frac{\partial F}{\partial\rho}\frac{d\rho}{dn}, \tag{9}$$
this normal being taken outwards.

Poincaré takes the equation of the geoid S^* in the form
$$\rho=\rho_0+\left(\frac{d\rho}{dn}\right)_0\zeta, \tag{10}$$
ζ being a function of μ and ν, whose square is supposed to be negligible, while $(d\rho/dn)_0$ indicates what the function H becomes for $\rho=\rho_0$. It follows that in passing from the point M of the ellipsoid S to the point M^* of the geoid S^* the increment of the coordinate ρ, along the line ρ through M, is $(d\rho/dn)_0\zeta$.

Let us call V_0 the external Newtonian potential in the first ellipsoidal hypothesis (in which the rigorous expression of V_0 is well known); U the potential of the centrifugal force, that is, $\frac{1}{2}\omega^2(x^2+y^2)$; $W_0=V_0+U$ the total terrestrial potential, always on the hypothesis

(4). Calling, instead, V and W respectively the Newtonian potential and the total one on the hypothesis (10), Poincaré writes

$$W = W_0+\delta V = V_0+\delta V+U,$$

where δV is clearly of the order of ζ.

On the surface S, we have, by hypothesis,

$$(W)_S = (V_0)_S+(U)_S = C, \tag{11}$$

C being a constant.

On the surface S^*, we must have

$$(W)_{S^*} = (V_0)_{S^*}+(\delta V)_{S^*}+(U)_{S^*} = C+\delta C. \tag{12}$$

Poincaré supposes $\delta C = 0$; but, strictly speaking, C depends upon the total mass M, the rotation ω, and the surface S; it accordingly varies in passing from S to S^*.

Omitting only terms in ζ^2, we have

$$(V_0)_{S^*} = (V_0)_S+\zeta\frac{\partial V_0}{\partial\rho_0}\left(\frac{d\rho}{dn}\right)_0,$$

$$(U)_{S^*} = (U)_S+\zeta\frac{\partial U}{\partial\rho_0}\left(\frac{d\rho}{dn}\right)_0,$$

$$(\delta V)_{S^*} = (\delta V)_S.$$

To the same approximation,† we then deduce from (11) and (12)

$$\zeta\frac{\partial W_0}{\partial\rho_0}\left(\frac{d\rho}{dn}\right)_0+(\delta V)_S = \delta C.$$

For the value of gravity g on S, on the hypothesis (4), we have

$$g = -\frac{\partial W_0}{\partial\rho_0}\left(\frac{d\rho}{dn}\right)_0;$$

and we may then write

$$g\zeta = (\delta V)_S-\delta C. \tag{13}$$

Let us pass to the expression of gravity g^* on S^*, on the second hypothesis (10). The element of arc of the line ρ, issuing from the point M of S, is $H^{-1}\,d\rho$, and consequently the component of g^* along this line is $H\partial W/\partial\rho$. Denoting by n^* the normal at the point M^* of S^*, taken outwards, and by (ρ, n^*) the angle between this normal and the above line ρ, we may then write

$$g^* = -\frac{1}{\cos(\rho, n^*)}\left(H\frac{\partial W}{\partial\rho}\right)_{S^*}.$$

† This, however, implies that equation (11) itself must subsist as far as the terms of the order of ζ^2, at least.

Poincaré implicitly supposes $\cos(\rho, n^*) = 1$; but this is not correct to the desired approximation. Indeed we have

$$\cos(\rho, n^*) = 1+\epsilon\zeta Q,$$

where Q is a function of μ and ν, linear and homogeneous with respect to the three quantities ζ, $\partial\zeta/\partial\mu$, $\partial\zeta/\partial\nu$; and ϵ may be called the *ellipticity* of our ellipsoid, putting

$$\epsilon = \frac{a+b-2c}{2a}.$$

It follows that Poincaré, in fact, neglects terms of the order of $\epsilon\zeta$, which are much greater than the terms of the order of ζ^2. On the other hand, if the terms of the order of $\epsilon\zeta$ were retained, the problem, on account of the intricate expression of Q, would become very arduous and not reducible to the third boundary-value problem of the potential theory. *We shall neglect, then, the quantities of the order of $\epsilon\zeta$.* To this approximation, we have simply

$$g^* = -\left(H\frac{\partial W}{\partial \rho}\right)_{S^*}. \tag{14}$$

But

$$(H)_{S^*} = \left(\frac{d\rho}{dn}\right)_0\left(1+\zeta\frac{\partial H}{\partial\rho_0}\right),$$

$$\left(\frac{\partial W}{\partial\rho}\right)_{S^*} = \frac{\partial W_0}{\partial\rho_0}+\frac{\partial^2 W_0}{\partial\rho_0^2}\left(\frac{d\rho}{dn}\right)_0\zeta+\frac{\partial}{\partial\rho_0}(\delta V);$$

so that

$$g^* = -\left(\frac{d\rho}{dn}\right)_0\left\{\frac{\partial W_0}{\partial\rho_0}+\frac{\partial W_0}{\partial\rho_0}\frac{\partial H}{\partial\rho_0}\zeta+\frac{\partial^2 W_0}{\partial\rho_0^2}\left(\frac{d\rho}{dn}\right)_0\zeta+\frac{\partial}{\partial\rho_0}(\delta V)\right\}.$$

Taking account of (13), this equation may be written

$$g^*-g = -\left(\frac{d\rho}{dn}\right)_0^2\frac{1}{g}\,\frac{\partial^2 W_0}{\partial\rho_0^2}\{(\delta V)_S-\delta C\}-\left(\frac{d\rho}{dn}\right)_0\frac{\partial}{\partial\rho_0}(\delta V)-$$

$$-\frac{\partial W_0}{\partial\rho_0}\frac{\partial H}{\partial\rho_0}\left(\frac{d\rho}{dn}\right)_0\frac{1}{g}\{(\delta V)_S-\delta C\}. \tag{15}$$

In the equation which is found by Poincaré [loc. cit., p. 37, formula (1)], the last term on the right hand of our (15) is absent. This depends on the fact that Poincaré does not consider the variation of $d\rho/dn$ in passing from S to S^*; but this variation can never be neglected. On the other hand, in (15) there are terms of the order of $\epsilon\zeta$ which are superfluous, since terms of the same order have already been neglected.

Indeed we have

$$H_0 = \left(\frac{d\rho}{dn}\right)_0 = \frac{a}{\rho_0}+\ldots, \qquad \left(\frac{\partial H}{\partial \rho}\right)_{\rho=\rho_0} = \frac{1}{a}-\frac{a}{\rho_0^2}+\ldots,$$

where the terms omitted are of the order of the ellipticity ϵ. It follows that equation (15), save the terms in $\epsilon\zeta$, may be written

$$g-g^* = \frac{a}{\rho_0}\frac{\partial}{\partial \rho_0}(\delta V)+\frac{a}{\rho_0}\frac{(\delta V)_S-\delta C}{g}\left\{\frac{a}{\rho_0}\frac{\partial^2 W_0}{\partial \rho_0^2}+\left(\frac{1}{a}-\frac{a}{\rho_0^2}\right)\frac{\partial W_0}{\partial \rho_0}\right\}. \quad (16)$$

Thus the question is reduced to finding a function δV harmonic outside the ellipsoid S, regular and null at infinity and satisfying, on the boundary, a condition of the type

$$\frac{d}{dn}(\delta V)+p(\delta V)_S = k,$$

where p and k are given functions of μ and ν (third boundary-value problem). This problem, in the case of an ellipsoid, is soluble (if at all) by means of Lamé functions.

4. But it is easy to show that, to the present approximation, the problem reduces to a spherical one. Let us pass indeed from the confocal coordinates ρ, μ, ν to the polar coordinates r, θ, ϕ. We have, neglecting terms in ϵ,

$$r = \sqrt{\rho^2-\alpha^2}+\ldots, \qquad \frac{\partial W_0}{\partial \rho} = \frac{\partial W_0}{\partial r}\frac{\rho}{\sqrt{\rho^2-\alpha^2}}+\ldots,$$

$$\frac{\partial^2 W_0}{\partial \rho_0^2} = \frac{\partial^2 W_0}{\partial r^2}\frac{\rho^2}{\rho^2-\alpha^2}+\frac{1}{\sqrt{\rho^2-\alpha^2}}\left(1-\frac{\rho^2}{\rho^2-\alpha^2}\right)\frac{\partial W_0}{\partial r};$$

and then

$$\frac{\partial W_0}{\partial \rho_0} = \left(\frac{\partial W_0}{\partial r}\right)_{r=a}\frac{\rho_0}{a}, \qquad \frac{\partial^2 W_0}{\partial \rho_0^2} = \frac{\rho_0^2}{a^2}\left(\frac{\partial^2 W_0}{\partial r^2}\right)_{r=a}-\left(\frac{1}{a}-\frac{\rho_0^2}{a^3}\right)\left(\frac{\partial W_0}{\partial r}\right)_{r=a}.$$

Consequently equation (16), neglecting the terms in $\epsilon\zeta$, becomes

$$g-g^* = \frac{1}{g}\{(\delta V)_{r=a}-\delta C\}\left(\frac{\partial^2 W_0}{\partial r^2}\right)_{r=a}+\left(\frac{\partial}{\partial r}(\delta V)\right)_{r=a}. \quad (17)$$

We are thus led to the sphere. Equation (17) is the one obtained in the ordinary theory of Stokes. To the approximation which we can really attain, it is, then, quite useless to take as reference surface an ellipsoid of three unequal axes and to employ confocal coordinates. It is sufficient to move from (1) for passing to (3). On neglecting, as

Poincaré does, the terms in $\alpha\beta$, we clearly have, to this approximation,

$$\zeta = -\beta au; \tag{18}$$

and putting
$$\delta V = \beta f V^*, \qquad \delta C = \beta C^*, \tag{19}$$

where f is the constant of gravitation, equation (17) becomes

$$g - g^* = \beta f\left(\frac{\partial V^*}{\partial r}\right)_{r=a} + \frac{\beta f}{g}\left(V^* - \frac{C^*}{f}\right)\left(\frac{\partial^2 W_0}{\partial r^2}\right)_{r=a}. \tag{20}$$

But we have, omitting terms in α,

$$W_0 = \frac{fM}{r} + \frac{\omega^2 a^5}{3r^3}P_2 + \tfrac{1}{2}\omega^2 r^2 \sin^2\theta, \qquad \left(\frac{\partial^2 W_0}{\partial r^2}\right)_{r=a} = \frac{2fM}{a^3} + \tfrac{2}{3}\omega^2(1+5P_2),$$

$$g = \frac{fM}{a^2} + \tfrac{1}{3}\omega^2 a(5P_2 - 2),$$

where P_2 is the Legendre function of the second degree in $\cos\theta$.

Consequently, equation (20) becomes, on the sphere of radius a,

$$\frac{\partial V^*}{\partial r} + pV^* = k, \tag{21}$$

where
$$p = \frac{2}{a}\frac{fM/a^2 + \frac{1}{3}\omega^2 a + \frac{5}{3}\omega^2 a P_2}{fM/a^2 - \frac{2}{3}\omega^2 a + \frac{5}{3}\omega^2 a P_2}, \qquad k = \frac{g - g^*}{\beta f} + \frac{C^*}{f}p. \tag{22}$$

5. Equation (21) is more general than that which is found in Stokes's theory, and it reduces to that one when terms in $\beta\omega^2$ are neglected. But these terms could not be rejected, for instance, for a planet turning, *caeteris paribus*, with an angular velocity ten times that of the Earth. A further step may be taken by integrating equation (21) even in the case of non-slow rotations.

Let us put, for brevity,
$$\gamma_0 = \frac{fM}{a^2}. \tag{23}$$

We get
$$p = \frac{2}{a}\left\{1 + \frac{\omega^2 a}{\gamma_0} + \frac{\omega^4 a^2}{\gamma_0^2}(2 - 5P_2) + \ldots\right\}. \tag{24}$$

Suppose that $g^* - g$ admits an expansion in a series of Laplace's functions

$$g^* - g = \beta \sum_{m=0}^{\infty} G_m, \tag{25}$$

G_m being a known surface spherical harmonic of order m.

Putting V^* in the form

$$V^* = v_0 + v_1\omega^2 + v_2\omega^4 + \ldots,$$

it follows that

$$\left.\begin{aligned}
\frac{\partial v_0}{\partial r}+\frac{2}{a}v_0 &= \frac{2C^*}{af}-\frac{1}{f}\sum_{m=0}^{\infty} G_m,\\
\frac{\partial v_1}{\partial r}+\frac{2}{a}v_1 &= \frac{2C^*}{\gamma_0 f}-\frac{2}{\gamma_0}v_0,\\
\frac{\partial v_2}{\partial r}+\frac{2}{a}v_2 &= \frac{2aC^*}{3\gamma_0^2 f}(2-5P_2)-\frac{2a}{3\gamma_0^2}(2-5P_2)v_0-\frac{2}{\gamma_0}v_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}. \qquad (26)$$

Let us put

$$v_0 = \sum_{m=0}^{\infty}\frac{Y_m^{(0)}}{r^{m+1}}, \qquad v_1 = \sum_{m=0}^{\infty}\frac{Y_m^{(1)}}{r^{m+1}}, \qquad v_2 = \sum_{m=0}^{\infty}\frac{Y_m^{(2)}}{r^{m+1}},\ldots.$$

From the first of (26) follows the condition

$$G_1 = 0,$$

which has been indicated in § 2. From the standpoint of differential equations of type (21), this condition may be explained by the fact that equation (21) admits of a solution uniquely determined when the function p is never positive; but, in our case, p is always positive and then the problem is not always possible.

It follows further, from the first of (26), that

$$Y_0^{(0)} = \frac{a}{f}(2C^*-aG_0), \qquad Y_m^{(0)} = \frac{a^{m+2}}{(m-1)f}G_m \quad (m > 1);$$

that is to say, that functions $Y_m^{(0)}$ are uniquely determined, save, for the moment, $Y_1^{(0)}$.

Similarly, from the second equation of (26) it is possible to deduce functions $Y_m^{(1)}$, except $Y_1^{(1)}$; but since in the right-hand member of the above equation the surface spherical harmonic of the first order must be nul, this condition enables us to determine $Y_1^{(0)}$, which had not been determined by the first of (26). It is found, in this case, that

$$Y_1^{(0)} = 0.$$

Continuing in this way, it is not difficult to see that all the functions $Y_m^{(i)}$ may be determined. The constant C^* is as yet arbitrary; but on imposing the condition that the mass M of the Earth must be unchanged, we have

$$Y_0^{(0)}+Y_0^{(1)}\omega^2+Y_0^{(2)}\omega^4+\ldots = 0,$$

and this condition enables us to determine C^*. Consequently, from the relation

$$agu = C^* - fV^*,$$

the required function u follows at once.

6. In general, it seems impossible to make further progress with the problem without knowing the law of development of the function g^*. If this development be given, one may attempt to find a spheroidal surface S^*, on which gravity coincides with g^*. Some research has been done on these lines.†

† See Mineo, 'On the expansion of the Earth's gravity . . .': *Quart. J. of Math.* (Oxford), 1 (1930), 116–21; and also Gulotta, 'Sullo sviluppo rigoroso in serie di funzioni sferiche del potenziale esterno e della gravità superficiale d'un pianeta sferoidico non di rotazione': *Rendiconti dei Lincei,* vol. xi, Roma 1930.

INTEGRALS FOR THE PRODUCT OF TWO BESSEL FUNCTIONS

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1. Introduction

1.1. Well-known particular cases of the type of formula we consider in this paper are*

$$\pi J_\mu(z)J_\nu(z) = 2\int\limits_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2z\cos\theta)\cos(\mu-\nu)\theta\,d\theta,$$

$$K_\mu(z)K_\nu(z) = 2\int\limits_0^\infty K_{\mu+\nu}(2z\cosh t)\cosh(\mu-\nu)t\,dt,$$

$$K_\mu(iz)K_\mu(-iz) = 2\int\limits_0^\infty K_0(2z\sinh t)\cosh 2\mu t\,dt,$$

$$\pi J_0(Z)J_0(z) = \int\limits_0^\pi J_0\{\surd(Z^2+z^2-2Zz\cos\phi)\}\,d\phi.$$

We consider the products $K_\mu(Z)K_\nu(z)$, $J_\mu(Z)J_\nu(z)$, where $Z \neq z$, and the chief results which we obtain are

(i) $K_\mu(Z)K_\nu(z)$

$$= \int\limits_{-\infty}^\infty e^{-(\mu-\nu)t}\left\{\frac{Ze^t+ze^{-t}}{Ze^{-t}+ze^t}\right\}^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\surd(Z^2+z^2+2Zz\cosh 2t)\}\,dt,$$

which is valid when $R(Z)$ and $R(z)$ are both positive, and also, with certain restrictions, and with a proper choice of the path of integration, when $R(Z)$ and $R(z)$ are zero.

We give the proof in considerable detail for the three cases (a) when Z and z are both on the real axis; (b), (c) when Z and z are on the imaginary axis and on the same side, or on opposite sides, of the origin.

* Watson, *Theory of Bessel Functions*, 150, 440, 444 (1), 367.

(ii) If X and x are real, $X > x$, and $R(\mu-\nu) < \frac{1}{2}$,

$$2\pi J_\mu(X)J_\nu(x)$$

$$= \int_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)}\{\cos\nu\pi J_{\mu+\nu}(\varpi)-\sin\nu\pi Y_{\mu+\nu}(\varpi)\}\,d\phi\,-$$

$$-2\sin\nu\pi\int_{0}^{\infty} e^{-\nu u}\left(\frac{X+xe^{u}}{X+xe^{-u}}\right)^{\frac{1}{2}(\mu+\nu)}\{\cos\nu\pi J_{\mu+\nu}(\Omega)-\sin\nu\pi Y_{\mu+\nu}(\Omega)\}\,du,$$

where

$$\varpi = \sqrt{(X^2+x^2-2Xx\cos\phi)},$$
$$\Omega = \sqrt{(X^2+x^2+2Xx\cosh u)}.$$

In this integral expression for $J_\mu(X)J_\nu(x)$, the argument of the Bessel functions in the integrand does not vanish, so that it is generally possible to make use of asymptotic values; in the last part of the paper we give another integral expression, (7.22), more compact, but with a less manageable integrand.

This last formula (7.22) is equivalent to the theorem that

$$\sum_{m=-\infty}^{\infty}\frac{J_{\mu-m}(X)J_{\nu+m}(x)}{X^{\mu-m}x^{\nu+m}}e^{m\phi i}$$

is the Fourier series, uniformly convergent with respect to ϕ in $(-\pi,\pi)$ when $R(\mu+\nu) > 0$, of

$$e^{\frac{1}{2}(\mu-\nu)\phi i}\left\{\frac{2\cos\frac{1}{2}\phi}{X^2e^{\frac{1}{2}\phi i}+x^2e^{-\frac{1}{2}\phi i}}\right\}^{\frac{1}{2}(\mu+\nu)}J_{\mu+\nu}(\Phi),$$

where

$$\Phi = \sqrt{\{(2\cos\tfrac{1}{2}\phi)(X^2e^{\frac{1}{2}\phi i}+x^2e^{-\frac{1}{2}\phi i})\}}.$$

The application of Fourier's integral theorem to the formula (7.22) leads to what Watson calls Ramanujan's extraordinary integrals (Watson, loc. cit. 449), so that (7.22) and Ramanujan's formula may be regarded as Fourier transforms.

1.2. *Notation.* Throughout we use X, x to denote *positive* numbers; Xi, xi are numbers of argument $\frac{1}{2}\pi$, and $-Xi$, $-xi$ are numbers of argument $-\frac{1}{2}\pi$.

2. The expression of $K_\mu(X)K_\nu(x)$ as an integral

From the well-known formula*

$$K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh t-\nu t}\,dt \qquad (|\arg z| < \tfrac{1}{2}\pi), \tag{2.1}$$

* Watson, loc. cit. 182 (7).

we have, by direct multiplication,

$$K_\mu(X)K_\nu(x) = \lim_{R\to\infty} \tfrac{1}{4} \int_{-R}^{R} e^{-X\cosh t-\mu t}\,dt \int_{-R}^{R} e^{-x\cosh u-\nu u}\,du. \tag{2.2}$$

If S_R denotes the square bounded by $u = \pm R$, $t = \pm R$ we may write this as

$$4K_\mu(X)K_\nu(x) = \lim_{R\to\infty} \iint_{S_R} e^{-X\cosh t-x\cosh u}\,e^{-\mu t-\nu u}\,du dt.$$

Put $t = T+U$, $u = T-U$ and let Q_R denote the square bounded by $T = \pm R$, $U = \pm R$. Then, since the integrand is positive, and

$$Q_{R/\surd 2} \subset S_R \subset Q_{R\surd 2} \subset S_{2R},$$

$$\begin{aligned} 4K_\mu(X)K_\nu(x) &= \lim_{R\to\infty} \iint_{Q_R} e^{-X\cosh t-x\cosh u}\,e^{-\mu t-\nu u}\,du dt \\ &= 2\lim_{R\to\infty} \iint_{Q_R} e^{-(\mu+\nu)T-(\mu-\nu)U} e^{-X\cosh(T+U)-x\cosh(T-U)}\,dU dT. \end{aligned}$$

Now consider, for a fixed value of T,

$$y = \exp\{-X\cosh(T+U)-x\cosh(T-U)\}.$$

The greatest value of y comes when U is given in terms of T by

$$(x+X)\tanh U = (x-X)\tanh T,$$

and is $$\exp\left[-(x+X)\cosh T\left\{1-\left(\frac{x-X}{x+X}\right)^2\tanh^2 T\right\}^{\frac{1}{2}}\right].$$

It is now easy to see that, for large R,

$$\int_{-R}^{R} e^{-(\mu-\nu)U}\,dU \int_{R}^{\infty} e^{-(\mu+\nu)T} e^{-X\cosh(T+U)-x\cosh(T-U)}\,dT = o(1), \tag{2.3}$$

together with a similar result when the T-range of integration is $(-\infty, -R)$. Hence*

$$K_\mu(X)K_\nu(x) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-(\mu-\nu)U}\,dU \int_{-\infty}^{\infty} e^{-(\mu+\nu)T} e^{-X\cosh(T+U)-x\cosh(T-U)}\,dT. \tag{2.4}$$

Now write, in the T integral,

$$(X+x)\cosh U = \lambda\cosh\psi, \qquad (X-x)\sinh U = \lambda\sinh\psi,$$

* Cf. Watson, loc. cit. 440. Here, as in Watson, all the integrals are absolutely convergent and an appeal to general theorems can be made. The above direct justification of the various transformations in the simple case of absolute convergence may serve to mark the main steps in the later work where, with non-absolute convergence, arithmetical detail obscures the argument.

where

$$\lambda = +\sqrt{(X^2+x^2+2Xx\cosh 2U)} = +\sqrt{\{(Xe^U+xe^{-U})(Xe^{-U}+xe^U)\}},$$

and
$$\exp\psi = +\sqrt{\{(Xe^U+xe^{-U})/(Xe^{-U}+xe^U)\}}.$$

We thus get

$$K_\mu(X)K_\nu(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-(\mu-\nu)U}\,dU \int_{-\infty}^{\infty} e^{-(\mu+\nu)T-\lambda\cosh(T+\psi)}\,dT$$

$$= \int_{-\infty}^{\infty} e^{-(\mu-\nu)U+(\mu+\nu)\psi} K_{\mu+\nu}(\lambda)\,dU,$$

on using (2.1). That is to say,

$$K_\mu(X)K_\nu(x) = \int_{-\infty}^{\infty} e^{-(\mu-\nu)U}\left\{\frac{Xe^U+xe^{-U}}{Xe^{-U}+xe^U}\right\}^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}(\lambda)\,dU, \tag{2.5}$$

where
$$\lambda = +\sqrt{\{(Xe^U+xe^{-U})(Xe^{-U}+xe^U)\}}.$$

3. The expression of $K_\mu(iX)K_\nu(ix)$ as an integral

3.1. We begin with the necessary modification* of (2.1), namely,

$$K_\mu(iX) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-iX\cosh t-\mu t}\,dt \qquad (-1 < R(\mu) < 1). \tag{3.11}$$

We proceed as in § 2 (with the same notation) and obtain

$$4K_\mu(iX)K_\nu(ix) = \lim_{R\to\infty} \iint_{S_R} e^{-iX\cosh t-ix\cosh u}\, e^{-\mu t-\nu u}\,dudt. \tag{3.12}$$

In this double integral we may replace squares S_R by squares Q_R [bounded by $t+u = \pm 2R$, $t-u = \pm 2R$] if we can show that

$$\iint e^{-iX\cosh t-ix\cosh u}\, e^{\pm\mu t\pm\nu u}\,dudt$$

taken over each of the triangles

(i) sides $t = 0$, $u = R$, $t+u = 2R$,

(ii) sides $u = 0$, $t = R$, $t+u = 2R$,

tends to zero as R tends to infinity. This we can do if μ and ν are suitably restricted.

* Watson, loc. cit. 180 (11); Dixon and Ferrar, *Quart J. of Math.* (Oxford), 1 (1930), 236.

Suppose that $|R(\mu)| < \frac{1}{2}$, $|R(\nu)| < \frac{1}{2}$. Then*

$$\int\limits_{R}^{2R-t} e^{-ix\cosh u}\, e^{\pm\nu u}\, du$$

$$= \left[e^{-ix\cosh u}\frac{ie^{\pm\nu u}}{x\sinh u}\right]_R^{2R-t} - \frac{i}{x}\int\limits_{R}^{2R-t} e^{-ix\cosh u}\,\frac{d}{du}\left(\frac{e^{\pm\nu u}}{\sinh u}\right) du$$

$$= o\,(e^{-\frac{1}{2}R}) + o\,(e^{-\frac{1}{2}R}),$$

and the double integral over the triangle (i) is less in modulus than

$$o\,(e^{-\frac{1}{2}R})\int\limits_{0}^{R} e^{\frac{1}{2}t}\, dt = o\,(1).$$

Similarly, the integral over the triangle (ii) is $o(1)$.

Hence, in (3.12), we may replace S_R by Q_R and, on writing $t = T+U$, $u = T-U$, obtain

$$4K_\mu(iX)K_\nu(ix)$$
$$= 2\lim_{R\to\infty}\int\limits_{-R}^{R}\int\limits_{-R}^{R} e^{-iX\cosh(T+U)-ix\cosh(T-U)}e^{-(\mu+\nu)T-(\mu-\nu)U}\, dU dT. \tag{3.13}$$

We now show that, with the further restrictions on μ and ν, $|R(\mu)| < \frac{1}{4}$ and $|R(\nu)| < \frac{1}{4}$, (2.4) is true when we replace X, x by iX, ix.

3.2. Consider

$$\int\limits_{R}^{\infty} e^{-(\mu+\nu)T}e^{-iX\cosh(T+U)-ix\cosh(T-U)}\, dT, \tag{3.21}$$

where $|R(\mu)| < \frac{1}{4}$ and $|R(\nu)| < \frac{1}{4}$, for any fixed value of U lying between $-R$ and R. Write

$$X\cosh(T+U) + x\cosh(T-U) = \alpha,$$

so that $$e^{-i\alpha} = \frac{i}{X\sinh(T+U)+x\sinh(T-U)}\,\frac{d}{dT}(e^{-i\alpha}).$$

On integrating by parts, (3.21) may be written as

$$\left[\frac{o\,(e^{\frac{1}{2}T})}{X\sinh(T+U)+x\sinh(T-U)}\right]_R^\infty -$$

$$-i\int\limits_{R}^{\infty} e^{-i\alpha}\,\frac{d}{dT}\left\{\frac{e^{-(\mu+\nu)T}}{X\sinh(T+U)+x\sinh(T-U)}\right\} dT.$$

* We used the same method in a former paper, *Quart. J. of Math.* (Oxford), 1 (1930), 139.

For any value* of U, if T be large enough to make $(X+x)\tanh T$ greater than $|X-x|$,

$$X\sinh(T+U)+x\sinh(T-U) \geqslant (X+x)\sinh T\left\{1-\left(\frac{X-x}{X+x}\right)^2\coth^2 T\right\}^{\frac{1}{2}},$$

and

$$\frac{X\cosh(T+U)+x\cosh(T-U)}{X\sinh(T+U)+x\sinh(T-U)} \leqslant \frac{(X+x)+|X-x|\tanh T}{(X+x)\tanh T-|X-x|}.$$

It is now easy to see that (3.21) is $o(e^{-\frac{1}{2}R})$, so that

$$\left|\int\limits_{-R}^{R} e^{-(\mu-\nu)U}\,dU \int\limits_{R}^{\infty} e^{-(\mu+\nu)T}e^{-iX\cosh(T+U)-ix\cosh(T-U)}\,dT\right|$$

$$< \int\limits_{-R}^{R} e^{\frac{1}{2}|U|}o(e^{-\frac{1}{2}R})\,dU = o(1).$$

There is a similar inequality when the limits of the T integral are $(-\infty, -R)$.

3.3. Hence, from (3.13), we have

$$K_\mu(iX)K_\nu(ix)$$

$$= \tfrac{1}{2}\lim_{R\to\infty} \int\limits_{-R}^{R} e^{-(\mu-\nu)U}\,dU \int\limits_{-\infty}^{\infty} e^{-(\mu+\nu)T}e^{-iX\cosh(T+U)-ix\cosh(T-U)}\,dT. \quad (3.31)$$

This is the result of (2.4) with iX, ix for X, x and there is no difficulty in carrying out the remaining transformations of § 2. We thus obtain our second key formula

$$K_\mu(iX)K_\nu(ix) = \int\limits_{-\infty}^{\infty} e^{-(\mu-\nu)U}\left\{\frac{Xe^U+xe^{-U}}{Xe^{-U}+xe^U}\right\}^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}(i\lambda)\,dU, \quad (3.32)$$

valid,† by analytical continuation, when $|R(\mu-\nu)| < \frac{3}{2}$.

4. The expression of $K_\mu(iX)K_\nu(-ix)$ as an integral; $X > x$

4.1. First assume that $|R(\mu)| < \frac{1}{4}$, $|R(\nu)| < \frac{1}{4}$. With slight changes, purely arithmetical in character, the work of § 3 as far as (3.31) may be carried out with $-x$ instead of x. This gives

* The first result is obtained by finding the minimum of the left-hand side, *qua* function of U: the second result uses $0 \leqslant |\tanh U| \leqslant 1$.

† It is fairly easy to see that the result holds when $|R(\mu-\nu)| < \frac{1}{2}$; the technique of integration by parts, here very laborious, is necessary to the proof of the convergence of the integral for the wider range of values.

$$K_\mu(iX)K_\nu(-ix)$$
$$= \tfrac{1}{2}\int\limits_{-\infty}^{\infty} e^{-(\mu-\nu)U}\,dU \int\limits_{-\infty}^{\infty} e^{-(\mu+\nu)T}e^{-iX\cosh(T+U)+ix\cosh(T-U)}\,dT. \qquad (4.11)$$

On the other hand the transformations whereby (3.32) follows from (3.31) require considerable modification.

We may write (4.11) in the form

$$2K_\mu(iX)K_\nu(-ix) = A+B,$$

where

$$A = \int\limits_{0}^{\infty} e^{-(\mu-\nu)U}\,dU \int\limits_{-\infty}^{\infty} e^{-(\mu+\nu)T}e^{-i(X-x)\cosh T\cosh U-i(X+x)\sinh T\sinh U}\,dT$$

and B is obtained by putting $-U$ for U in A.

4.2. Let $e^{2\alpha} = (X/x)$. Then

$$(X-x)^2-4Xx\sinh^2 U \geqslant 0 \quad \text{when } 0 \leqslant U \leqslant \alpha.$$

We may, when $0 \leqslant U \leqslant \alpha$, write

$$(X-x)\cosh U = \lambda_1\cosh\psi_1, \qquad (X+x)\sinh U = \lambda_1\sinh\psi_1,$$

where λ_1 is real and positive (or zero), and λ_1, ψ_1 are given by

$$\lambda_1 = +\sqrt{\{(X-x)^2-4Xx\sinh^2 U\}}, \qquad e^{2\psi_1} = \frac{Xe^U-xe^{-U}}{Xe^{-U}-xe^U}. \qquad (4.21)$$

If A_1, B_1 are the parts of A, B given by $0 \leqslant U \leqslant \alpha$, then

$$A_1 = \int\limits_{0}^{\alpha} e^{-(\mu-\nu)U}\,dU \int\limits_{-\infty}^{\infty} e^{-i\lambda_1\cosh(T+\psi_1)}e^{-(\mu+\nu)T}\,dT$$
$$= 2\int\limits_{0}^{\alpha} e^{(\mu+\nu)\psi_1-(\mu-\nu)U}K_{\mu+\nu}(i\lambda_1)\,dU, \qquad (4.22)$$

the last step by means of the formula (3.11). Similarly,

$$B_1 = 2\int\limits_{0}^{\alpha} e^{-(\mu+\nu)\psi_1+(\mu-\nu)U}K_{\mu+\nu}(i\lambda_1)\,dU. \qquad (4.23)$$

4.3. When $U \geqslant \alpha$, $4Xx\sinh^2 U-(X-x)^2 \geqslant 0$, and we may write

$$(X-x)\cosh U = \lambda_2\sinh\psi_2, \qquad (X+x)\sinh U = \lambda_2\cosh\psi_2,$$

where λ_2 is real and positive (or zero) and λ_2, ψ_2 are given by

$$\lambda_2 = +\sqrt{\{4Xx\sinh^2 U-(X-x)^2\}}, \qquad e^{2\psi_2} = \frac{Xe^U-xe^{-U}}{xe^U-Xe^{-U}}.$$

Note that ψ_2 is real for the values of U under consideration.

If A_2, B_2 are the parts of A, B given by $U \geqslant \alpha$, then

$$A_2 = \int_{\alpha}^{\infty} e^{-(\mu-\nu)U}\, dU \int_{-\infty}^{\infty} e^{-i\lambda_2 \sinh(T+\psi_2)} e^{-(\mu+\nu)T}\, dT,$$

which may be written as*

$$A_2 = 2\int_{\alpha}^{\infty} e^{-(\mu-\nu)U+(\mu+\nu)\psi_2}\, e^{\frac{1}{2}(\mu+\nu)\pi i} K_{\mu+\nu}(\lambda_2)\, dU. \tag{4.31}$$

Similarly,

$$B_2 = 2\int_{\alpha}^{\infty} e^{(\mu-\nu)U-(\mu+\nu)\psi_2}\, e^{-\frac{1}{2}(\mu+\nu)\pi i} K_{\mu+\nu}(\lambda_2)\, dU. \tag{4.32}$$

4.4. We now combine A_1 and A_2 into a single integral. Consider

$$2\int_{0}^{\infty} e^{-(\mu-\nu)U}\left(\frac{Xe^{U}-xe^{-U}}{Xe^{-U}-xe^{U}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(Xe^{U}-xe^{-U})(xe^{U}-Xe^{-U})}\}\, dU, \tag{4.41}$$

where the path of integration is the real axis indented *upwards* at the point $\frac{1}{2}\log(X/x)$ and the determinations of the multiple-valued functions are fixed by the conventions that when $U = 0$

$$\arg(Xe^{U}-xe^{-U}) = 0, \qquad \arg(Xe^{-U}-xe^{U}) = 0,$$
$$\arg(xe^{U}-Xe^{-U}) = \pi.$$

For $0 \leqslant U < \alpha$ the integrand in (4.41) is equivalent to the integrand in (4.22), namely, the A_1 integral. For $U > \alpha$,

$$\arg(Xe^{U}-xe^{-U}) = 0, \qquad \arg(Xe^{-U}-xe^{U}) = -\pi,$$
$$\arg(xe^{U}-Xe^{-U}) = 0,$$

when $U = \frac{1}{2}\log(X/x)$ is passed by an indentation above the real axis. Thus, for $U > \alpha$ the integrand in (4.41) is equivalent to the integrand in (4.31), namely, the A_2 integral. Hence (4.41) is equal to A_1+A_2, and so is equal to A. The expressions B_1, B_2 differ from A_1, A_2 in the signs of μ, ν only. Accordingly, on making a slight change in the form (4.41), we have

$$K_{\mu}(iX)K_{\nu}(-ix)$$
$$= \int^{\infty} e^{2\nu U}\left(\frac{X-xe^{-2U}}{X-xe^{2U}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(Xe^{U}-xe^{-U})(xe^{U}-Xe^{-U})}\}\, dU+$$
$$+\int_{0}^{\infty} e^{-2\nu U}\left(\frac{X-xe^{-2U}}{X-xe^{2U}}\right)^{-\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{\sqrt{(Xe^{U}-xe^{-U})(xe^{U}-Xe^{-U})}\}\, dU, \tag{4.42}$$

* Compare Watson, loc. cit. 182 (10).

where the path is indented above $\frac{1}{2}\log(X/x)$ and the determinations of the functions are fixed by the foregoing conventions.

In this form all restrictions on μ and ν may be removed.

5. Increasing the argument by $\frac{1}{2}\pi$

5.1. In a former paper* we developed in detail a process which we called 'increasing the argument by $\frac{1}{2}\pi$'. We now make use of the same process as before in order to find, what was already known in the particular case then considered, an integral expression for $J_\mu(X)J_\nu(x)$. A comparison of that work with what follows will throw some light on our present method. The setting out of our argument is, however, independent of our former work.

5.2. In the integrals of (4.42) change the variable from U to s, where

$$e^U = s+\surd(1+s^2), \tag{5.21}$$

i.e. $\sinh U = s$ and $dU = ds/\surd(1+s^2)$. The corresponding s path of integration will then have an indentation above the real axis of s. Consider now the contour formed by

(i) the arc $s = Re^{\theta i}$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, where R is large;

(ii) the real axis, indented upwards at $s = (X-x)/2\surd(Xx)$;

(iii) the imaginary axis indented to the right at $s = i$.

The latter indentation is necessary in order to keep the integrand a one-valued function of s, since $s = i$ is a branch point.

It is not difficult to show that when $|R(\mu-\nu)| < \frac{1}{2}$ the integrals over (i) tend to zero as $R \to \infty$, the integrands being those of (4.42) after the substitution (5.21). Hence, if $F(s)$ denote either integrand, we shall get

$$\int_0^\infty F(s)\,ds = i\int_0^\infty F(ti)\,dt,$$

where the indentations are duly taken into account. We may then, instead of (5.21), use the substitutions

$$e^U = it+\surd(1-t^2) \qquad (0 \leqslant t < 1),$$
$$e^U = it+i\surd(t^2-1) \qquad (t > 1),$$

to transform our integrals. If now we write $t = \cos\theta$ when $0 \leqslant t < 1$

* *Quart J. of Math.* (Oxford) 1 (1930), 122–45: in particular §§ 2.2, 4.2.

and $t = \cosh u$ when $t > 1$, these substitutions become

$$e^U = i\cos\theta+\sin\theta = ie^{-i\theta},$$
$$e^U = i\cosh u+i\sinh u = ie^{u},$$

respectively.

By these means we may write (4.42) as

$$K_\mu(iX)K_\nu(-ix)$$
$$= i\int\limits_0^{\frac{1}{2}\pi} e^{\nu\pi i-2\nu\theta i}\left(\frac{X+xe^{2i\theta}}{X+xe^{-2i\theta}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{i\sqrt{(X^2+x^2+2Xx\cos 2\theta)}\}\,d\theta +$$
$$+\int\limits_0^{\infty} e^{\nu\pi i+2\nu u}\left(\frac{X+xe^{-2u}}{X+xe^{2u}}\right)^{\frac{1}{2}(\mu+\nu)} K_{\mu+\nu}\{i\sqrt{(X^2+x^2+2Xx\cosh 2u)}\}\,du +$$

+two terms which are derived from these by writing $-\mu$, $-\nu$ for μ, ν.

$$= ie^{\nu\pi i}P_1+e^{\nu\pi i}Q_1+ie^{-\nu\pi i}P_2+e^{-\nu\pi i}Q_2, \tag{5.22}$$

say.

In the second part of our paper we use this form and not the form (4.42).

PART II

6. The expression of $J_\mu(X)J_\nu(x)$ as an integral; $X > x$

6.1. We now deduce from a combination of (5.22) and (3.32) the formula for $J_\mu(X)J_\nu(x)$ given in § 1. The notation used is that of § 5 (5.22).

We have proved, (5.22),

$$K_\mu(iX)K_\nu(-ix) = e^{\nu\pi i}(iP_1+Q_1)+e^{-\nu\pi i}(iP_2+Q_2), \tag{6.11}$$

and, in (3.32),

$$K_\mu(iX)K_\nu(ix) = Q_1+Q_2. \tag{6.12}$$

Both formulae are true if $|R(\mu-\nu)| < \frac{1}{2}$.

It is known that

$$J_\nu(x)+iY_\nu(x) = H_\nu^{(1)}(x) = \frac{2}{\pi i}e^{-\frac{1}{2}\nu\pi i}K_\nu(-ix), \tag{6.13}$$

$$J_\nu(x)-iY_\nu(x) = -\frac{2}{\pi i}e^{\frac{1}{2}\nu\pi i}K_\nu(ix), \tag{6.14}$$

and so, from (6.11) and (6.12),

$$2K_\mu(iX)J_\nu(x) = \frac{2}{\pi}\{e^{\frac{1}{2}\nu\pi i}P_1+e^{-\frac{1}{2}\nu\pi i}P_2\}+\frac{2}{\pi i}\{e^{-\frac{1}{2}\nu\pi i}-e^{\frac{1}{2}\nu\pi i}\}Q_2. \tag{6.15}$$

6.2. In order to facilitate the algebraical manipulations which follow we introduce the notations

$$\lambda = \sqrt{(X^2+x^2+2Xx\cos 2\theta)}, \qquad \Lambda = \sqrt{(X^2+x^2+2Xx\cosh 2u)},$$

$$P_{1j} = \int_0^{\frac{1}{2}\pi} e^{-2\nu\theta i}\left(\frac{X+xe^{2i\theta}}{X+xe^{-2i\theta}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\lambda)\,d\theta,$$

$$P_{2j} = \int_0^{\frac{1}{2}\pi} e^{2\nu\theta i}\left(\frac{X+xe^{2i\theta}}{X+xe^{-2i\theta}}\right)^{-\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\lambda)\,d\theta,$$

$$Q_{2j} = \int_0^{\infty} e^{-2\nu u}\left(\frac{X+xe^{2u}}{X+xe^{-2u}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\Lambda)\,du.$$

The corresponding integrals with the Bessel function $Y_{\mu+\nu}$ replacing $J_{\mu+\nu}$ are denoted by P_{1y}, P_{2y}, Q_{2y} respectively. Now [compare (5.22)] P_1 is an integral which involves $K_{\mu+\nu}(i\lambda)$ and, on using (6.14), we see that

$$P_{1j}-iP_{1y} = -\frac{2}{\pi i}e^{\frac{1}{2}(\mu+\nu)\pi i}P_1. \tag{6.21}$$

The same type of relation holds also for P_2 and Q_2.

By (6.14) and (6.15) we get

$$\{J_\mu(X)-iY_\mu(X)\}J_\nu(x) = -\frac{2}{\pi i}e^{\frac{1}{2}\mu\pi i}K_\mu(iX)J_\nu(x)$$

$$= \frac{2ie^{\frac{1}{2}\mu\pi i}}{\pi^2}[e^{\frac{1}{2}\nu\pi i}P_1+e^{-\frac{1}{2}\nu\pi i}P_2+i(e^{\frac{1}{2}\nu\pi i}-e^{-\frac{1}{2}\nu\pi i})Q_2]. \tag{6.22}$$

This, on using (6.21) and its analogues, becomes

$$\frac{1}{\pi}[(P_{1j}-iP_{1y})+e^{-2\nu\pi i}(P_{2j}-iP_{2y})+i(1-e^{-2\nu\pi i})(Q_{2j}-iQ_{2y})].$$

Now P_{1j}, P_{2j} are conjugate complex numbers; so also are P_{1y}, P_{2y}: the numbers Q_{2j}, Q_{2y} are real.

Adding our previous result to the conjugate expression for $\{J_\mu(X)+iY_\mu(X)\}J_\nu(x)$, we obtain

$$\begin{aligned}2\pi J_\mu(X)J_\nu(x) &= P_{1j}(1+e^{2\nu\pi i})+P_{2j}(1+e^{-2\nu\pi i})+\\ &\quad +iP_{1y}(e^{2\nu\pi i}-1)+iP_{2y}(1-e^{-2\nu\pi i})+\\ &\quad +Q_{2j}(-2\sin 2\nu\pi)+2Q_{2y}(1-\cos 2\nu\pi).\end{aligned} \tag{6.23}$$

6.3. A suitable grouping of the terms of (6.23) simplifies that result considerably:

$$2e^{\nu\pi i}P_{1j} = 2\int_0^{\frac{1}{2}\pi} e^{\nu\pi i-2\nu\theta i}\left(\frac{X+xe^{2i\theta}}{X+xe^{-2i\theta}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\lambda)\,d\theta.$$

Put $2\theta = \pi-\phi$, and write $\varpi = \sqrt{(X^2+x^2-2Xx\cos\phi)}$. We get

$$2e^{\nu\pi i}P_{1j} = \int_0^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\varpi)\,d\phi. \tag{6.31}$$

Similarly, on using the substitution $2\theta = \pi+\phi$,

$$2e^{-\nu\pi i}P_{2j} = \int_{-\pi}^{0} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\varpi)\,d\phi. \tag{6.32}$$

Hence,

$$P_{1j}(1+e^{2\nu\pi i})+P_{2j}(1+e^{-2\nu\pi i})$$

$$= \cos\nu\pi \int_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\varpi)\,d\phi. \tag{6.33}$$

The terms in P_{1y} and P_{2y} of (6.23) combine in a similar fashion to give

$$-\sin\nu\pi \int_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)} Y_{\mu+\nu}(\varpi)\,d\phi. \tag{6.34}$$

Hence (6.23) may be written as

$$2\pi J_\mu(X)J_\nu(x) \qquad (X > x) \tag{6.35}$$

$$= \int_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+\nu)} \{\cos\nu\pi J_{\mu+\nu}(\varpi)-\sin\nu\pi Y_{\mu+\nu}(\varpi)\}\,d\phi -$$

$$-2\sin\nu\pi \int_0^{\infty} e^{-\nu u}\left(\frac{X+xe^{u}}{X+xe^{-u}}\right)^{\frac{1}{2}(\mu+\nu)} \{\cos\nu\pi J_{\mu+\nu}(\Omega)-\sin\nu\pi Y_{\mu+\nu}(\Omega)\}\,du,$$

where

$$\varpi = \sqrt{(X^2+x^2-2Xx\cos\phi)}, \qquad \Omega = \sqrt{(X^2+x^2+2Xx\cosh u)}.$$

We have proved the formula on the assumption that $|R(\mu-\nu)| < \frac{1}{2}$; actually $R(\mu-\nu) < \frac{1}{2}$ is sufficient for the truth of the formula itself and is necessary for the convergence of the last integral.

6.4. *Special forms*; $X > x$.

If $\nu = n$, an integer, (6.35) reduces to

$$(-)^{|n|}2\pi J_\mu(X)J_n(x) = \int\limits_{-\pi}^{\pi} e^{n\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(\mu+n)} J_{\mu+n}(\varpi)\,d\phi. \tag{6.41}$$

If $\mu = m$, an integer, (6.35) reduces to

$$(-)^{|m|}2\pi J_m(X)J_\nu(x) = \int\limits_{-\pi}^{\pi} e^{\nu\phi i}\left(\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right)^{\frac{1}{2}(m+\nu)} J_{-m-\nu}(\varpi)\,d\phi\, -$$

$$-2\sin\nu\pi \int\limits_{0}^{\infty} e^{-\nu u}\left(\frac{X+xe^{u}}{X+xe^{-u}}\right)^{\frac{1}{2}(\mu+\nu)} J_{-m-\nu}(\Omega)\,du. \tag{6.42}$$

The result (6.41) is not new, being merely the integral form of Graf's expansion*

$$J_\mu(\varpi)\left\{\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right\}^{\frac{1}{2}\mu} = \sum_{n=-\infty}^{\infty} J_{\mu+n}(X)J_n(x)e^{ni\phi}. \tag{6.43}$$

If we multiply this by $\exp(-ni\phi)$ and integrate, we get

$$(-)^n J_{\mu+n}(X)J_{-n}(x) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} J_\mu(\varpi)\left\{\frac{X-xe^{-i\phi}}{X-xe^{i\phi}}\right\}^{\frac{1}{2}\mu} e^{-ni\phi}\,d\phi, \tag{6.44}$$

which is a variant of (6.41).

7. A formula derived by direct expansion

7.1. The particular formula

$$J_\mu(x)J_\nu(x) = \frac{2}{\pi}\int\limits_{0}^{\frac{1}{2}\pi} J_{\mu+\nu}(2x\cos\theta)\cos(\mu-\nu)\theta\,d\theta \tag{7.11}$$

is well known. It is given by Watson (loc. cit. 150) and is proved by a method which applies equally well to the expression of $J_\mu(X)J_\nu(x)$ as an integral.

7.2. Assume throughout that $R(\mu+\nu) > -1$. If $R(p+q) > -1$, then†

$$\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} = \frac{2^{p+q}}{\pi}\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{p+q}\theta\, e^{i(p-q)\theta}\,d\theta. \tag{7.21}$$

* Watson, loc. cit. 359; unless μ is an integer or zero it requires $|x| < |X|$.

† Watson, loc. cit. 150; or Whittaker and Watson, *Modern Analysis*, 263.

Now, by direct calculation, we have

$$J_\mu(Xz)J_\nu(xz) = \sum_{r=0}^{\infty} \frac{(-)^r(\frac{1}{2}Xz)^{2r+\mu}}{r!\Gamma(\mu+r+1)} \sum_{s=0}^{\infty} \frac{(-)^s(\frac{1}{2}xz)^{2s+\nu}}{s!\Gamma(\nu+s+1)}$$

$$= X^\mu x^\nu \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}z)^{2m+\mu+\nu}}{m!\Gamma(m+\mu+\nu+1)} \sum_{r+s=m} X^{2r}x^{2s}\frac{m!}{r!s!} \frac{\Gamma(m+\mu+\nu+1)}{\Gamma(\mu+r+1)\Gamma(\nu+s+1)}.$$

On using (7.21), this becomes

$$\frac{X^\mu x^\nu}{\pi} \sum_{m=0}^{\infty} \frac{(-\frac{1}{2})^m z^{2m+\mu+\nu}}{m!\Gamma(m+\mu+\nu+1)} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^{m+\mu+\nu}\theta(X^2e^{i\theta}+x^2e^{-i\theta})^m e^{i(\mu-\nu)\theta}\, d\theta.$$

Hence

$$J_\mu(Xz)J_\nu(xz) = \frac{X^\mu x^\nu}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{i(\mu-\nu)\theta}(\lambda_1/\lambda_2)^{\mu+\nu} J_{\mu+\nu}(z\lambda_1\lambda_2)\, d\theta, \quad (7.22)$$

where $\lambda_1 = +\sqrt{(e^{i\theta}+e^{-i\theta})}$, $\lambda_2 = (X^2e^{i\theta}+x^2e^{-i\theta})^{\frac{1}{2}}$. It will be seen that the determination of the two-valued function λ_2 does not affect the value of the integrand.

What relation (7.22) bears to the Graf integral (6.44) is by no means obvious at sight. We find what this relation is by considering the contour integral form of (7.22).

7.3. *An alternative form.* Consider

$$\frac{1}{2\pi i} \int_{|t|=X/x} t^{-\nu-1}\left\{\frac{X+xt}{X+xt^{-1}}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}\{\sqrt{(X+xt)(X+xt^{-1})}\}\, dt, \quad (7.31)$$

where the argument of t varies from $-\pi$ to π, and the determination of the multiple-valued integrand is fixed by the hypothesis

$$(X+xt)^{\frac{1}{2}} \text{ has the value } +\sqrt{(2X)} \text{ when } t = X/x.$$

On writing $t = (X/x)\exp(2i\theta)$, (7.31) becomes

$$\frac{X^\mu x^\nu}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{-2\nu\theta i}\left\{\frac{1+e^{2i\theta}}{X^2+x^2e^{-2i\theta}}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}\{\sqrt{(1+e^{2i\theta})(X^2+x^2e^{-2i\theta})}\}\, d\theta,$$

where, at $\theta = 0$, $(1+e^{2i\theta})^{\frac{1}{2}}$ is $\sqrt{2}$.

A slight change of form enables us to identify this with (7.22) when $z = 1$, and so (7.31) is an expression for $J_\mu(X)J_\nu(x)$.

7.4. *Its connexion with Graf's integral.*

Suppose now that $X > x$. Take the circle $t = (X/x)e^{i\phi}$, $-\pi \leqslant \phi \leqslant \pi$, and make a small indentation at $t = -(X/x)$.

Consider the integral of (7.31) taken round a contour Γ which

(i) begins at $t = (\delta - X/x)e^{-\pi i}$,

(ii) ends at $t = (\delta - X/x)e^{\pi i}$,

(iii) follows the small indentation near the two end points and, for the rest, follows the circle $|t| = X/x$.

With our hypothesis, $R(\mu+\nu) > -1$, the limit of such an integral as $\delta \to 0$ is the integral (7.31). By introducing the small indentation we have excluded the branch point at $t = -(X/x)$.

Join the end points of Γ to the point $t = -1$ by two parallel lines and draw the unit circle to complete a contour lying wholly within Γ. The integral round Γ is equal to the integrals along the parallel lines (each with its appropriate value of $t^{-\nu-1}$) together with the integral round the unit circle.

On making $\delta \to 0$ after this transformation has been effected, we obtain

$$J_\mu(X)J_\nu(x) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{-\nu\theta i}\left\{\frac{X+xe^{i\theta}}{X+xe^{-i\theta}}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(\varpi_1)\,d\theta - $$

$$-\frac{\sin\nu\pi}{\pi}\int\limits_{1}^{X/x} \rho^{-\nu-1}\left\{\frac{X-x\rho}{X-(x/\rho)}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(A)\,d\rho, \quad (7.41)$$

where $\varpi_1 = \sqrt{(X^2+x^2+2Xx\cos\theta)}$, $A = \sqrt{\{(X-x\rho)(X-x/\rho)\}}$.

When ν is an integer the second integral does not appear. A little manipulation* enables one to identify the present result for $\nu = n$ with the formula (6.41).

8. A symmetrical formula for $J_\mu(X)J_\nu(x)$

We conclude by stating the formula

$$2\pi i J_\mu(X)J_\nu(x) = \frac{e^{\nu\pi i}\sin\mu\pi}{\sin(\mu+\nu)\pi}\int\limits_{1}^{(0+)} z^{-\nu-1}\left\{\frac{X+xz}{X+x/z}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(Z)\,dz + $$

$$+\frac{e^{\mu\pi i}\sin\nu\pi}{\sin(\mu+\nu)\pi}\int\limits_{1}^{(0+)} z^{-\mu-1}\left\{\frac{x+Xz}{x+X/z}\right\}^{\frac{1}{2}(\mu+\nu)} J_{\mu+\nu}(Z)\,dz,$$

where $Z = \sqrt{\{X^2+x^2+Xx(z+z^{-1})\}}$ and has zero argument when $z = 1$.

* In (7.41), when $0 < \theta < \pi$ put $\pi-\theta = \phi$, when $-\pi < \theta < 0$ put $-\pi-\theta = \phi$.

To prove this we start from*

$$\frac{\Gamma(\nu+1)J_\mu(Xz)J_\nu(xz)}{(\frac{1}{2}Xz)^\mu(\frac{1}{2}xz)^\nu} = \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{1}{2}Xz)^{2m} {}_2F_1(-m, -m-\mu; \nu+1; x^2/X^2)}{\Gamma(m+1)\Gamma(m+\mu+1)}$$

and replace the hypergeometric function by what is in effect a Pochhammer loop integral.

The transformation, in the form we require, can be obtained without much difficulty by first considering the integral of

$$z^{\alpha-1}(1+\lambda z)^{\beta-1}(\lambda+z)^{\gamma-1} \qquad (\lambda < 1)$$

taken round the circle with centre at $z = 0$ and radius unity, starting from the point $z = 1$. With certain limitations on α, β, γ we can transform the contour into the real axis from 1 to 0, 0 to $-\lambda$, and back again, and obtain

$$i\int_0^{2\pi} e^{\alpha\theta i}(1+\lambda e^{\theta i})^{\beta-1}(\lambda+e^{\theta i})^{\gamma-1}\,d\theta$$
$$= (e^{2(\alpha+\gamma)\pi i}-1)\int_0^1 x^{\alpha-1}(1+\lambda x)^{\beta-1}(\lambda+x)^{\gamma-1}\,dx +$$
$$+e^{\alpha\pi i}(1-e^{2\gamma\pi i})\int_0^\lambda x^{\alpha-1}(1-\lambda x)^{\beta-1}(\lambda-x)^{\gamma-1}\,dx.$$

In this the last integral on the right-hand side is an integral expression of

$$\lambda^{\alpha+\gamma-1}\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\,{}_2F_1(1-\beta, \alpha; \alpha+\gamma; \lambda^2),$$

and, if the first integral be replaced by a loop integral round the origin, the restrictions on α, β, γ may be removed.

The symmetry of the formula when μ, ν and also X, x are interchanged is more apparent than real. If $X > x$ the point $z = -x/X$ lies inside the unit circle; it is a branch point of the second integrand but not of the first. Hence, in such a case, the first integral is simply another way of writing an integral taken round the unit circle, while the second is essentially a loop integral whose loop must cross the real axis between the points 0 and $-x/X$.

* Watson, loc. cit. 148 (2).

THE ZETA-FUNCTION OF RIEMANN; FURTHER DEVELOPMENTS OF VAN DER CORPUT'S METHOD

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1. In two recent papers in this Journal Titchmarsh* has given an account of van der Corput's method in the analytic theory of numbers and, applying his simplified form of the method to the theory of the Riemann zeta-function, he proved that

$$\zeta(\tfrac{1}{2}+it) = O\big(t^{\frac{27}{164}}\big).$$

He also gave a short proof of the well-known theorem that

$$\zeta(\sigma+it) = O(t^{1/(4Q-2)}\log t),$$

where $Q = 2^{q-1}$, on each of the lines $\sigma = 1-(q+1)/(4Q-2)$, for $q = 2, 3, 4,\ldots$, a theorem which, in a more general form, is due to van der Corput and Koksma.†

Going back to van der Corput's original method of exponent systems‡ and using various simplifications of it, we are able to give the two new results

$$\zeta(\tfrac{1}{2}+it) = O\big(t^{\frac{229}{1392}}\big)$$

and

$$\zeta(\sigma+it) = O\left(t^{\frac{1}{4Q-2}\left\{\frac{240Qq-16Q+128}{240Qq-15Q+128}\right\}}\right),$$

where $Q = 2^{q-1}$, on each of the lines $\sigma = 1-(q+1)/(4Q-2)$, for $q = 3, 4,\ldots$.

1.1. In his explanation of the method Titchmarsh§ uses the symbols A and B to denote the two processes contained in it. It may be pointed out that, in this paper, the processes A, B, $AA\ldots(= A^q)$ and $AA\ldots B(= A^qB)$ are represented by Theorems 1, 2, 3 and 4 respectively.

2. We begin with some lemmas.

2.1. Lemma 1. *Let $f(x)$ be a real function and let $H > 0$. Then*

$$\left|\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)}\right| = O\left(\frac{b-a}{\sqrt{H}}\right)+O\left(\frac{b-a}{H}\sum_{h=1}^{H-1}|U_h|\right)^{\frac{1}{2}}+O(H), \qquad (1)$$

* Titchmarsh (**1**) and (**2**). See list of references at the end.

† See Titchmarsh (**1**), 161.

‡ van der Corput (**5**).

§ Titchmarsh (**2**), 313.

where U_h is such that

$$\Big|\sum_{a\leqslant n\leqslant b-h} e^{2\pi i\{f(n+h)-f(n)\}}\Big| < |U_h|$$

for $1 \leqslant h < a-b$.

We may suppose a and b to be integers, since the right-hand side of (1) is unaltered if we replace a and b by the integers nearest to them. Also, we may suppose that $2 \leqslant H < b-a$, for the first term on the right of (1) is $O(b-a)$ if $H < 2$ and the third term on the right is $O(b-a)$ if $H \geqslant b-a$, in both of which cases the theorem is obviously true. Finally we may suppose H to be an integer, since the right-hand side of (1) is unaltered if we replace H by the integer nearest to it.

The lemma will now be proved if we show that

$$\Big|\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)}\Big| = O\Big(\frac{b-a}{\sqrt{H}}\Big)+O\Big(\frac{b-a}{H}\sum_{h=1}^{H-1}\Big|\sum_{a\leqslant n\leqslant b-h} e^{2\pi i\{f(n+h)-f(n)\}}\Big|\Big)^{\frac{1}{2}},$$

where a, b, and H are integers and $2 \leqslant H < b-a$. This is proved by Titchmarsh* and we shall not reproduce the proof here.

2.2. Lemma 2.† *If $f(x)$ is real and has continuous first and second differential coefficients, and $f''(x) \geqslant r$ (or $\leqslant -r$) throughout the interval (a, b), then*

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O\Big\{\frac{|f'(b)-f'(a)|}{\sqrt{r}}\Big\}+O\Big(\frac{1}{\sqrt{r}}\Big).$$

2.3. Lemma 3. *Let $f(x)$ be a real function with continuous differential coefficients of the first four orders, and $f''(x) < 0$ in the interval (a, b), and let*

$$m_2 \leqslant |f''(x)| < Am_2, \qquad |f'''(x)| < Am_3, \qquad |f^{(\mathrm{IV})}(x)| < Am_4,$$

where $m_3^2 = m_2 m_4$. Let $f'(b) = \alpha$, $f'(a) = \beta$, and n_ν be such that $f'(n_\nu) = \nu (\alpha \leqslant \nu \leqslant \beta)$. Then

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = e^{-\frac{1}{4}\pi i}\sum_{\alpha\leqslant\nu\leqslant\beta}\frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}}+O(m_2^{-\frac{1}{2}})+$$
$$+O[\log\{2+(b-a)m_2\}]+O\{(b-a)m_3^{\frac{1}{3}}\}. \quad (1)$$

This lemma differs from the corresponding one in Titchmarsh‡ only in the final error term in (1).

* Titchmarsh (**1**), Lemma 3.

† Titchmarsh (**1**), Theorem 1.

‡ Titchmarsh (**2**), Theorem 4.

Exactly as in his proof we can show that

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = \sum_{\alpha+1<\nu<\beta-1} \int\limits_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx + O(m_2^{-\frac{1}{2}}) + O\{\log(\beta-\alpha+2)\}, \tag{2}$$

and we shall therefore start from this result.

The sum on the right-hand side of (2) is empty unless $\beta-\alpha > 2$, so that we now suppose that this condition is satisfied. We write

$$\int\limits_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx = \int\limits_a^{n_\nu-\delta} + \int\limits_{n_\nu-\delta}^{n_\nu+\delta} + \int\limits_{n_\nu+\delta}^{b} = J_1+J_2+J_3.$$

Then

$$J_3 = \int\limits_{n_\nu+\delta}^{b} \frac{de^{2\pi i\{f(x)-\nu x\}}}{2\pi i\{f'(x)-\nu\}}$$

$$= O\left\{\frac{1}{\nu-f'(n_\nu+\delta)}\right\} = O\left\{\frac{1}{\delta|f''(n_\nu+\theta\delta)|}\right\} = O\left(\frac{1}{\delta m_2}\right),$$

and a similar result holds for J_1. Also

$$J_2 = \int\limits_{-\delta}^{\delta} e^{2\pi i\{f(n_\nu+t)-\nu n_\nu-\nu t\}}\,dt$$

$$= \int\limits_{-\delta}^{\delta} e^{2\pi i\{f(n_\nu)-\nu n_\nu+\frac{1}{2}t^2f''(n_\nu)+\frac{1}{6}t^3f'''(n_\nu)+\frac{1}{24}t^4f^{(\mathrm{iv})}(n_\nu+\theta t)\}}\,dt$$

$$= e^{2\pi i\{f(n_\nu)-\nu n_\nu\}} \int\limits_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)+\frac{1}{3}\pi i t^3 f'''(n_\nu)}\{1+O(t^4m_4)\}\,dt$$

$$= e^{2\pi i\{f(n_\nu)-\nu n_\nu\}} \int\limits_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)+\frac{1}{3}\pi i t^3 f'''(n_\nu)}\,dt + O(\delta^5 m_4).$$

Putting $\frac{1}{3}\pi i f'''(n_\nu) = \lambda$, we have

$$\int\limits_{-\delta}^{\delta} e^{\pi i t^2 f''(n_\nu)+\frac{1}{3}\pi i t^3 f'''(n_\nu)}\,dt = 2\int\limits_0^{\delta} e^{\pi i t^2 f''(n_\nu)}\left\{1+\sum_{r=1}^{\infty}\frac{(\lambda t^3)^{2r}}{(2r)!}\right\}dt,$$

and, since $f''(n_\nu) < 0$, this equals

$$\frac{1}{(\pi|f''(n_\nu)|)^{\frac{1}{2}}}\int\limits_0^{\pi\delta^2|f''(n_\nu)|}\frac{e^{-iu}}{\sqrt{u}}\,du + \sum_{r=1}^{\infty}\frac{1}{(2r)!}\frac{\lambda^{2r}}{(\pi|f''(n_\nu)|)^{3r+\frac{1}{2}}}\int\limits_0^{\pi\delta^2|f''(n_\nu)|} e^{-iu}u^{3r-\frac{1}{2}}\,du.$$

The first term of this equals

$$\frac{e^{-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} - \frac{1}{(\pi|f''(n_\nu)|)^{\frac{1}{2}}}\int\limits_{\pi\delta^2|f''(n_\nu)|}^{\infty}\frac{e^{-iu}}{\sqrt{u}}\,du = \frac{e^{-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}} + O\left(\frac{1}{\delta m_2}\right).$$

The other terms are

$$\sum_{1}^{\infty}\frac{1}{(2r)!}\,\frac{\lambda^{2r}\,\cdot}{(\pi|f''(n_\nu)|)^{3r+\frac{1}{2}}}[O(\delta^2|f''(n_\nu)|)^{3r-\frac{1}{2}}]$$

$$= O\left[\frac{1}{\delta m_2}\left\{\sum_{1}^{\infty}\frac{1}{(2r)!}(\delta^3\lambda)^{2r}\right\}\right]$$

$$= O\left[\frac{1}{\delta m_2}e^{A\delta^3 m_3}\right].$$

Altogether we have

$$\int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx = \frac{e^{2\pi i\{f''(n_\nu)-\nu n_\nu\}-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}}+O\left(\frac{1}{\delta m_2}\right)+O(\delta^5 m_4)+O\left(\frac{1}{\delta m_2}e^{A\delta^3 m_3}\right).$$

The first two O-terms are of the same order if $\delta = (m_2 m_4)^{-\frac{1}{6}} = m_3^{-\frac{1}{3}}$; and then the third O-term is also of this order. If this value of δ is such that

$$a \leqslant n_\nu-\delta, \qquad n_\nu+\delta \leqslant b, \tag{3}$$

we obtain

$$\int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx = \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}-\frac{1}{4}\pi i}}{|f''(n_\nu)|^{\frac{1}{2}}}+O(m_3^{\frac{1}{3}}m_2^{-1}).$$

If (3) is satisfied for all values of ν such that $\alpha+1 < \nu < \beta-1$, we obtain, on summing with respect to ν,

$$\sum_{\alpha+1<\nu<\beta-1}\int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx$$

$$= e^{-\frac{1}{4}\pi i}\sum_{\alpha+1<\nu<\beta-1}\frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}}+O\{(\beta-\alpha)m_2^{-1}m_3^{\frac{1}{3}}\}. \tag{4}$$

The limits of the sum on the right may be replaced by (α,β) with error $O(m_2^{-\frac{1}{2}})$ and

$$\beta-\alpha = f'(a)-f'(b) = O\{(b-a)m_2\},$$

so that the last term in (4) is $O\{(b-a)m_3^{\frac{1}{3}}\}$. The lemma then follows from (2).

Suppose next that there are values of ν for which $n_\nu+\delta > b$. The argument proceeds as before except that we now obtain a term

$$\int_{-\delta}^{b-n_\nu} e^{\pi i t^2 f''(n_\nu)+\frac{1}{3}\pi i t^3 f'''(n_\nu)}\,dt$$

instead of the corresponding integral over $(-\delta,\delta)$. We therefore have to consider an additional error

$$\int\limits_{b-n_\nu}^{\delta} e^{\pi i t^2 f''(n_\nu)+\frac{1}{3}\pi i t^3 f'''(n_\nu)}\,dt$$

$$= \int\limits_{b-n_\nu}^{\delta} e^{\pi i t^2 f''(n_\nu)}\,dt + \sum_{1}^{\infty}\frac{\lambda^r}{r!}\int\limits_{b-n_\nu}^{\delta} e^{\pi i t^2 f''(n_\nu)} t^{3r}\,dt$$

$$= O\{m_2^{-1}(b-n_\nu)^{-1}\}+O\left\{\frac{1}{\delta m_2}e^{A\delta^3 m_3}\right\}$$

$$= O\{m_2^{-1}(b-n_\nu)^{-1}\}+O(m_3^{\frac{1}{3}}m_2^{-1}).$$

If $\psi(x)$ is the inverse function of $f'(x)$, so that

$$|\psi'(x)| = |f''(x)|^{-1} \geqslant m_2^{-1},$$

then $b-n_\nu = \psi(\alpha)-\psi(\nu) = (\alpha-\nu)\psi'(\xi) \geqslant (\nu-\alpha)/m_2$. Hence the sum of these terms is

$$O\left(\sum_{\alpha+1<\nu<\beta-1}\frac{1}{\nu-\alpha}\right)+O\{(\beta-\alpha)m_3^{\frac{1}{3}}m_2^{-1}\}$$

$$= O\{\log(\beta-\alpha)\}+O\{(\beta-\alpha)m_3^{\frac{1}{3}}m_2^{-1}\}$$

$$= O\{\log 2+(b-a)m_2\}+O\{(b-a)m_3^{\frac{1}{3}}\},$$

and the lemma again follows. A similar argument disposes of the case $n_\nu-\delta < a$.

2.4. LEMMA 4.* *Let r be an integer $\geqslant 5$, $0<a<b$, $f(x)$ a real function with differential coefficients of the first r orders, and $f''(x)<0$ in (a,b); $\alpha = f'(b)$, $\beta = f'(a)$, and n_ν such that $f'(n_\nu) = \nu$ $(\alpha \leqslant \nu \leqslant \beta)$. Put $\phi(\nu) = -f(n_\nu)+\nu n_\nu$ and let $0<\gamma<\frac{1}{2}$, $s = 1/\sigma > 0$, $y>0$, $\eta = y^\sigma$.*

Then there is a positive number $c<\frac{1}{2}$, depending only on r, γ, and s such that if

$$|f^{(p+1)}(n)-(-1)^p ys(s+1)\dots(s+p-1)n^{-s-p}| < cys(s+1)\dots(s+p-1)n^{-s-p}\dagger$$

for all n in (a,b) and all integral $p \geqslant 0$ and $\leqslant r-1$, then $\phi(\nu)$ has differential coefficients of the first r orders and

$$|\phi^{(p+1)}(\nu)-(-1)^p\eta\sigma(\sigma+1)\dots(\sigma+p-1)\nu^{-\sigma-p}| < \gamma\eta\sigma(\sigma+1)\dots(\sigma+p-1)\nu^{-\sigma-p}$$

for $\alpha \leqslant \nu \leqslant \beta$ and $0 \leqslant p \leqslant r-1$.

* van der Corput (5), Lemma 7, 56–8, where a proof is given.

† The expression $s(s+1)\dots(s+p-1)$ denotes 1 when $p = 0$.

3. DEFINITION.* *We shall say that the pair of absolute constants (k, l) is an exponent pair if*

$$0 \leqslant k \leqslant \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqslant l \leqslant 1$$

and, to every positive number s, there exist two numbers r and c depending only on s (r an integer $\geqslant 5$, $0 < c < \frac{1}{2}$) such that the inequality

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = O(z^k a^l) \tag{1}$$

holds with respect to s and u when the following conditions are satisfied:

$$u > 0, \qquad 1 \leqslant a < b < au, \qquad y > 0, \qquad z = ya^{-s} > 1; \tag{2}$$

$f(n)$ a real function with differential coefficients of the first r orders in (a, b) and

$$|f^{(p+1)}(n) - (-1)^p ys(s+1)...(s+p-1)n^{-s-p}|$$
$$< cys(s+1)...(s+p-1)n^{-s-p} \tag{3}$$

for $a \leqslant n \leqslant b$, $0 \leqslant p \leqslant r-1$.

It follows immediately that $(0, 1)$ is an exponent pair since

$$\Big|\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)}\Big| \leqslant b-a < au = uz^0 a$$

which is (1) with $k = 0$, $l = 1$.

The connexion of the theory of exponent pairs with the problem of the order of the zeta-function will be seen in Theorems 5 and 6.

3.1. THEOREM 1.* *If (κ, λ) is an exponent pair, then so is (k, l), where*

$$k = \frac{\kappa}{2(1+\kappa)}, \qquad l = \tfrac{1}{2} + \frac{\lambda}{2(1+\kappa)}.$$

We notice firstly that the inequalities

$$0 \leqslant k \leqslant \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqslant l \leqslant 1$$

follow from

$$0 \leqslant \kappa \leqslant \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqslant \lambda \leqslant 1.$$

We must show that to every positive number s we can find numbers r and c depending only on s (r an integer $\geqslant 5$, $0 < c < \frac{1}{2}$) such that the inequality

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = O(z^k a^l) \tag{1}$$

holds with respect to s and u when the following conditions are satisfied:

$$u > 0, \qquad 1 \leqslant a < b < au, \qquad y > 0, \qquad z = ya^{-s} > 1; \tag{2}$$

* Cf. van der Corput (**5**).

$f(n)$ is a real function with differential coefficients of the first r orders in (a,b) and

$$|f^{(p+1)}(n)-(-1)^p ys(s+1)...(s+p-1)n^{-s-p}| \\ < cys(s+1)...(s+p-1)n^{-s-p} \quad (3)$$

for $a \leqslant n \leqslant b$, $0 \leqslant p \leqslant r-1$.

We therefore suppose (2) and (3) to be true for values of r and c which we shall find in the course of the proof and we show that (1) then follows. Without loss of generality we can suppose $b-a \geqslant 1$, since otherwise the sum in (1) is $O(1)$. Let $\sigma = s+1$, $\tau = u$.

Since (κ, λ) is an exponent pair, there are two numbers ρ and γ (ρ an integer $\geqslant 5$, $0 < \gamma < \frac{1}{2}$) depending only on σ, and therefore only on s, such that the inequality

$$\sum_{\alpha \leqslant \nu \leqslant \beta} e^{2\pi i\,\phi(\nu)} = O(\zeta^\kappa \alpha^\lambda) \quad (4)$$

holds with respect to σ and τ (that is, with respect to s and u), when the following conditions are satisfied:

$$1 \leqslant \alpha < \beta < \alpha\tau, \qquad \eta > 0, \qquad \zeta = \eta\alpha^{-\sigma} > 1; \quad (5)$$

$\phi(\nu)$ a real function with differential coefficients of the first ρ orders in (α, β) and

$$|\phi^{(p+1)}(\nu)-(-1)^p \eta\sigma(\sigma+1)...(\sigma+p-1)\nu^{-\sigma-p}| \\ < \gamma\eta\sigma(\sigma+1)...(\sigma+p-1)\nu^{-\sigma-p} \quad (6)$$

for $\alpha \leqslant \nu \leqslant \beta$, $0 \leqslant p \leqslant \rho-1$.

We now choose $r = \rho+2$ and apply Lemma 1 to the sum in (1). Since $f(n)$ has differential coefficients of the first $\rho+2$ orders, so also has $\Delta_h(\nu) = f(\nu+h)-f(\nu)$ in the interval (α_h, β_h), where $\alpha_h = a$, $\beta_h = b-h$.

We have
$$\Delta_h(\nu) = \int_0^h f'(\nu+x)\,dx,$$

hence
$$\Delta_h^{(p+1)}(\nu) = \int_0^h f^{(p+2)}(\nu+x)\,dx$$

for $\alpha_h \leqslant \nu \leqslant \beta_h$, $0 \leqslant p \leqslant \rho-1$. Therefore, using (3), we have

$$\Delta_h^{(p+1)}(\nu) = (-1)^{p+1} ys(s+1)...(s+p) \int_0^h (1+\theta_1 c)(\nu+x)^{-s-p-1}\,dx, \quad (7)$$

where $|\theta_1| < 1$. We now choose c $(0 < c < \frac{1}{2})$ depending only on s, so that for all p $(0 \leqslant p \leqslant \rho-1)$ and all θ_1 and θ_2 $(|\theta_1| < 1,\ |\theta_2| < 1)$ we have
$$(1+\theta_1 c)(1+\theta_2 c)^{-s-p-1} = 1+\theta_3\gamma \quad (|\theta_3| < 1).$$

In our use of Lemma 1 we shall suppose $H < ac$. We then have $0 \leqslant x \leqslant h < H < ac \leqslant \nu c$, so that the integrand in (7)

$$\begin{aligned} &= (1+\theta_1 c)(\nu+x)^{-s-p-1} \\ &= (1+\theta_1 c)(1+\theta_2 c)^{-s-p-1}\nu^{-s-p-1} \\ &= (1+\theta_3 \gamma)\nu^{-s-p-1}. \end{aligned}$$

Hence from (7) we have ($|\theta_4| < 1$)

$$-\Delta_h^{(p+1)}(\nu) = (-1)^p(1+\theta_4\gamma)ys(s+1)...(s+p)h\nu^{-s-p-1}$$

for $\alpha_h \leqslant \nu \leqslant \beta_h$, $0 \leqslant p \leqslant \rho-1$.

If we now put $-\Delta_h(\nu) = \phi_h(\nu)$ and $ysh = \eta_h$ we obtain

$$\phi_h^{(p+1)}(\nu) = (-1)^p(1+\theta_4\gamma)\eta_h\sigma(\sigma+1)...(\sigma+p-1)\nu^{-\sigma-p}, \tag{8}$$

so that $\phi_h(\nu)$ satisfies the conditions (6), with $\eta = \eta_h = ysh$,

$$\alpha = \alpha_h = a, \qquad \beta = \beta_h = b-h.$$

Now, by Lemma 1, we have

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O\left(\frac{a}{\sqrt{H}}\right)+O\left\{\left(\frac{a}{H}\sum_{h=1}^{H-1} U_h\right)^{\frac{1}{2}}\right\}+O(H), \tag{9}$$

where $$\left|\sum_h\right| = \left|\sum_{\alpha_h\leqslant\nu\leqslant\beta_h} e^{2\pi i\,\phi_h(\nu)}\right| < \grave{U}_h.$$

We put $\zeta_h = \eta_h\alpha_h^{-\sigma} = ysha^{-s-1} = zsha^{-1}$ and we divide the sum on the right of (9) into two parts according as $h \leqslant a/sz$ or $h > a/sz$.

In the first case $\zeta_h \leqslant 1$ and from (8) we have

$$\begin{gathered} 0 < \tfrac{1}{2}\eta_h\nu^{-\sigma} < \phi_h'(\nu) < \tfrac{3}{2}\eta_h\nu^{-\sigma} \leqslant \tfrac{3}{2}\zeta_h < 2 \\ -\phi_h''(\nu) > \tfrac{1}{2}\eta_h\nu^{-\sigma-1} > A\eta_h\alpha_h^{-\sigma-1} > Azha^{-2} \\ (\alpha_h \leqslant \nu \leqslant \beta_h). \end{gathered}$$

Hence, applying Lemma 2, we have

$$\sum_h = O\{(zha^{-2})^{-\frac{1}{2}}\} = O(z^{-\frac{1}{2}}ah^{-\frac{1}{2}})$$

and $$\sum_{1\leqslant h\leqslant a/sz}\left|\sum_h\right| = O\left\{z^{-\frac{1}{2}}a\left(\frac{a}{z}\right)^{\frac{1}{2}}\right\}. \tag{10}$$

In the second case $\zeta_h > 1$, and then the conditions (5) are satisfied with $\alpha = \alpha_h$, $\beta = \beta_h$, $\eta = \eta_h$, and $\zeta = \zeta_h$. We have already proved that $\phi_h(\nu)$ satisfies (6), therefore (4) follows and we have

$$\sum_h = O(\zeta_h^\kappa\alpha_h^\lambda) = O(z^\kappa h^\kappa a^{\lambda-\kappa}).$$

Hence $$\sum_{a/sz<h\leqslant H-1}\left|\sum_h\right| \leqslant \sum_{1\leqslant h\leqslant H-1}\left|\sum_h\right| = O\{z^\kappa a^{\lambda-\kappa}H^{\kappa+1}\}. \tag{11}$$

Adding (10) and (11) and substituting in (9), we obtain

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = O\left(\frac{a}{\sqrt{H}}\right) + O\left\{\frac{a}{\sqrt{H}}\left(\frac{a}{z^2}\right)^{\frac{1}{4}}\right\} + O\{z^{\frac{1}{2}\kappa}a^{\frac{1}{2}(1+\lambda-\kappa)}H^{\frac{1}{2}\kappa}\} + O(H).$$

This has been proved under the supposition $H < ac$, but it remains true if $H \geqslant ac$, since then the modulus of the sum on the left does not exceed $b < ua \leqslant uH/c = O(H)$.

We now suppose $a \leqslant z^2$; then the second term on the right is less than the first and may be omitted. If $H = (a^{1-\lambda+\kappa}z^{-\kappa})^{1/(1+\kappa)}$, the third term is of the same order as the first and we obtain for this value of H

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = O\left\{a^{\frac{1}{2}+\frac{1}{2}\frac{\lambda}{1+\kappa}} z^{\frac{1}{2}\frac{\kappa}{1+\kappa}}\right\} + O\left\{a^{1-\frac{\lambda}{1+\kappa}} z^{-\frac{\kappa}{1+\kappa}}\right\}. \tag{12}$$

But since $0 \leqslant \kappa \leqslant \frac{1}{2}$, $\frac{1}{2} \leqslant \lambda \leqslant 1$, we have

$$\frac{1}{2} + \frac{1}{2}\frac{\lambda}{1+\kappa} \geqslant \frac{1}{2} + \frac{1}{6} = 1 - \frac{1}{3} \geqslant 1 - \frac{\lambda}{1+\kappa},$$

so that the second term on the right of (12) is less than the first and we have

$$\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} = O\left\{a^{\frac{1}{2}+\frac{1}{2}\frac{\lambda}{1+\kappa}} z^{\frac{1}{2}\frac{\kappa}{1+\kappa}}\right\} = O(z^k a^l).$$

This completes the proof of (1) for the case $a \leqslant z^2$. But it also holds for $a > z^2$; for we have, by (3),

$$0 < \tfrac{1}{2}yn^{-s} < f'(n) < 2yn^{-s} < 2z$$
$$-f''(n) > \tfrac{1}{2}ysn^{-s-1} > Aza^{-1}$$
$$(a \leqslant n \leqslant b)$$

and, applying Lemma 2, we obtain

$$\begin{aligned}
\sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} &= O\{z(za^{-1})^{-\frac{1}{2}}\} \\
&= O(z^{\frac{1}{2}}a^{\frac{1}{2}}) \\
&= O\left\{a^{\frac{1}{2}+\frac{1}{2}\frac{\lambda}{1+\kappa}} a^{-\frac{1}{2}\frac{\lambda}{1+\kappa}} z^{\frac{1}{2}}\right\} \\
&= O\left\{a^l z^{\frac{1}{2}\frac{1+\kappa-2\lambda}{1+\kappa}}\right\}, \qquad \text{since } a > z^2, \\
&= O\left\{a^l z^{\frac{1}{2}\frac{\kappa}{1+\kappa}}\right\}, \qquad \text{since } \lambda \geqslant \tfrac{1}{2}, \\
&= O(z^k a^l).
\end{aligned}$$

This completes the theorem.

3.2. THEOREM 2.* *If* (κ,λ) *is an exponent pair, then so is* (k,l), *where* $k=\lambda-\frac{1}{2}$, $l=\kappa+\frac{1}{2}$, *provided that* $2k+l\geqslant 1$.

We notice firstly that the inequalities

$$0\leqslant k\leqslant \tfrac{1}{2},\qquad \tfrac{1}{2}\leqslant l\leqslant 1$$

follow from

$$0\leqslant \kappa\leqslant \tfrac{1}{2},\qquad \tfrac{1}{2}\leqslant \lambda\leqslant 1.$$

We must show that, to every positive number s, we can find numbers r and c (r an integer $\geqslant 5$, $0<c<\frac{1}{2}$) which depend only on s, such that the inequality

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O(z^k a^l) \tag{1}$$

holds with respect to s and u when the following conditions are satisfied:

$$u>0,\qquad 1\leqslant a<b<au,\qquad y>0,\qquad z=ya^{-s}>1; \tag{2}$$

$f(n)$ is a real function with differential coefficients of the first r orders in (a,b) and

$$|f^{(p+1)}(n)-(-1)^p ys(s+1)...(s+p-1)n^{-s-p}| < cys(s+1)...(s+p-1)n^{-s-p} \tag{3}$$

for $a\leqslant n\leqslant b$, $0\leqslant p\leqslant r-1$.

We therefore suppose (2) and (3) to be true for values of r and c which we shall find in the course of the proof, and we show that (1) then follows.

Without loss of generality we can suppose $b>2^{1/s}$, since otherwise the left-hand side of (1) does not exceed b which is less than $2^{1/s}z^k a^l$.

From (3) we have

$$\left.\begin{aligned}\tfrac{1}{2}yn^{-s}<f'(n)<2yn^{-s}\\ \tfrac{1}{2}ysn^{-s-1}<-f''(n)<2ysn^{-s-1}\end{aligned}\right\}, \tag{4}$$

so that $f''(n)<0$, hence $0<f'(b)<f'(a)$.

We put $f'(b)=\alpha$, $f'(a)=\beta$, $1/s=\sigma$, $y^\sigma=\eta$, $\zeta=\eta\alpha^{-\sigma}$, $\tau=4u^s$. Then from (4) we have

$$\tfrac{1}{2}yb^{-s}<\alpha<2yb^{-s},$$

$$\tfrac{1}{2}z=\tfrac{1}{2}ya^{-s}<\beta<2z.$$

Hence $\alpha=O(z)$, $\qquad 2^{-\sigma}\zeta<b<2^\sigma\zeta$, $\qquad$ so that $1<\zeta=O(a)$, $\quad$ (5)

and $\dfrac{\beta}{\alpha}<\dfrac{2ya^{-s}}{\frac{1}{2}yb^{-s}}<4u^{-s}=\tau$, $\qquad$ so that $0<\alpha<\beta<\alpha\tau$.

The number n_ν is uniquely determined in $(\alpha\leqslant\nu\leqslant\beta)$ by the

* Cf. van der Corput (5).

relations $f'(n_\nu) = \nu$, $a \leqslant n_\nu \leqslant b$. We put $\phi(\nu) = -f(n_\nu)+\nu n_\nu$ in the interval (α, β).

We next suppose $\alpha \geqslant 1$. Then

$$\sigma > 0, \quad \tau > 0, \quad 1 \leqslant \alpha < \beta < \alpha\tau, \quad \eta > 0, \quad \zeta = \eta\alpha^{-\sigma} > 1. \tag{6}$$

Since (κ, λ) is an exponent pair, and since the conditions (6) are already satisfied, there are two numbers ρ and γ (ρ an integer $\geqslant 5$, $0 < \gamma < \frac{1}{2}$) depending only on σ and therefore only on s, such that if

$$|\phi^{(p+1)}(\nu) - (-1)^p \eta\sigma(\sigma+1)...(\sigma+p-1)\nu^{-\sigma-p}| \\ < \gamma\eta\sigma(\sigma+1)...(\sigma+p-1)\nu^{-\sigma-p} \tag{7}$$

$$(\alpha \leqslant \nu \leqslant \beta,\ 0 \leqslant p \leqslant \rho-1),$$

then, for every pair of numbers α_0, β_0 $(\alpha \leqslant \alpha_0 < \beta_0 \leqslant \beta)$, the inequality

$$\sum_{\alpha_0 \leqslant \nu \leqslant \beta_0} e^{-2\pi i\, \phi(\nu)} = O(\zeta^\kappa \alpha^\lambda) \tag{8}$$

holds with respect to σ and τ, that is, with respect to s and u.

The inequality (8) also holds in the case $\alpha < 1$; for then either the sum is empty or $\alpha\tau > \beta \geqslant \beta_0 \geqslant 1$ and the modulus of the sum $\leqslant \beta < \alpha\tau < \tau \leqslant \tau^{1+\lambda}\zeta^\kappa\alpha^\lambda$.

We now choose r equal to ρ. By Lemma 4, we can choose c depending only on r, γ, and s, that is, only on s, such that (7), and therefore (8), follow from (3). Hence, by (5), we have

$$\sum_{\alpha_0 \leqslant \nu \leqslant \beta_0} e^{2\pi i \{f(n_\nu) - \nu n_\nu\}} = O(\zeta^\kappa \alpha^\lambda) \\ = O(z^\lambda a^\kappa). \tag{8'}$$

From (3) we have $f'''(n) > 0$, so that $f''(n)$ is a monotonic function of n. Also

$$\frac{d\nu}{dn_\nu} = \frac{d f'(n_\nu)}{dn_\nu} = f''(n_\nu) < 0,$$

so that n_ν is a monotonic function of ν. Hence $|f''(n_\nu)|^{-\frac{1}{2}}$ is a monotonic function of ν and, by (4), it is $O(z^{-\frac{1}{2}}a^{\frac{1}{2}})$. Therefore, by partial summation, we obtain from this and (8')

$$\sum_{\alpha \leqslant \nu \leqslant \beta} \frac{e^{2\pi i \{f(n_\nu) - \nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}} = O(z^{\lambda-\frac{1}{2}} a^{\kappa+\frac{1}{2}}) \\ = O(z^k a^l). \tag{9}$$

From (3) again we have $(a \leqslant n \leqslant b)$ $Aza^{-1} < |f''(n)| < Aza^{-1}$, $|f'''(n)| < Aza^{-2}$, $|f^{(\mathrm{IV})}(n)| < Aza^{-3}$; $f^{(\mathrm{V})}(x)$ exists, so that $f^{(\mathrm{IV})}(x)$, $f'''(x)$, etc., are continuous; also $f''(x) < 0$. Hence the conditions of

Lemma 3 are satisfied with $m_2 = za^{-1}$, $m_3 = za^{-2}$, and applying it we obtain

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i\, f(n)} = e^{-\frac{1}{4}\pi i} \sum_{\alpha\leqslant \nu\leqslant\beta} \frac{e^{2\pi i\{f(n_\nu)-\nu n_\nu\}}}{|f''(n_\nu)|^{\frac{1}{2}}} + O(z^{-\frac{1}{2}}a^{\frac{1}{2}}) + $$
$$+O[\log(2+z)]+O(z^{\frac{1}{3}}a^{\frac{1}{3}})$$
$$= O(z^k a^l)+O(z^{\frac{1}{3}}a^{\frac{1}{3}}) \tag{10}$$

by (9), the logarithmic term being less than the second on the right and the term $O(z^{-\frac{1}{2}}a^{\frac{1}{2}})$ being less than the first since $l \geqslant \frac{1}{2}$.

If now $k \geqslant \frac{1}{3}$, the second term on the right of (10) is less than the first and the theorem follows. If $k < \frac{1}{3}$, we distinguish two cases according as $a \geqslant$ or $< z^{\frac{1-3k}{3l-1}}$.

1. Let $a \geqslant z^{\frac{1-3k}{3l-1}}$. Then

$$\begin{aligned} z^{\frac{1}{3}}a^{\frac{1}{3}} &= z^k a^{\frac{1}{3}} z^{\frac{1}{3}-k} \\ &\leqslant z^k a^{\frac{1}{3}} \Big(a^{\frac{3l-1}{1-3k}}\Big)^{\frac{1-3k}{3}} \\ &= z^k a^l, \end{aligned}$$

and again the second term on the right of (10) is less than the first.

2. Let $a < z^{\frac{1-3k}{3l-1}}$. Then

$$\begin{aligned} \sum_{a\leqslant n\leqslant b} e^{2\pi i\, f(n)} &= O(b-a) \\ &= O(a) \\ &= O(a^l a^{1-l}) \\ &= O\Big\{a^l \Big(z^{\frac{1-3k}{3l-1}}\Big)^{1-l}\Big\} \\ &= O(z^k a^l) \end{aligned}$$

since, from the hypothesis $2k+l \geqslant 1$, we have

$$\frac{1-3k}{3l-1}(1-l) \leqslant \frac{l-k}{2l-2k}\, 2k = k.$$

This completes the theorem.

3.3. Theorem 3. *If* (κ, λ) *is an exponent pair, then so is* (k_q, l_q), *where*

$$k_q = \frac{\kappa}{Q+2(Q-1)\kappa}, \qquad l_q = 1 - \frac{1-\lambda+q\kappa}{Q+2(Q-1)\kappa}$$

and q *is an integer* $\geqslant 1$, $Q = 2^q$.

In the case $q = 1$ the theorem reduces to Theorem 1. We prove the general case by Induction. We suppose the theorem true for

a particular value of q, that is, we suppose that (k_q, l_q) is an exponent pair. Then, by Theorem 1, so is the pair

$$\left(\frac{k_q}{2+2k_q},\ 1-\frac{1-l_q+k_q}{2+2k_q}\right),$$

i.e. $$\left(\frac{\kappa}{2Q+2(2Q-1)\kappa},\ 1-\frac{Q+(2Q-2)\kappa-Q+1-\lambda-(2Q-q-2)\kappa+\kappa}{2Q+2(2Q-1)\kappa}\right),$$

i.e. $$\left(\frac{\kappa}{2Q+2(2Q-1)\kappa},\ 1-\frac{1-\lambda+(q+1)\kappa}{2Q-2(2Q-1)\kappa}\right),$$

i.e. $$(k_{q+1}, l_{q+1}).$$

Thus the theorem follows by induction.

3.4. THEOREM 4. *If (κ, λ) is an exponent pair, then so is (k, l), where*

$$k = \tfrac{1}{2}-\frac{1-\lambda+q\kappa}{Q+2(Q-1)\kappa}, \qquad l = \tfrac{1}{2}+\frac{\kappa}{Q+2(Q-1)\kappa}$$

and q is an integer $\geqslant 1$, $Q = 2^q$.

By Theorem 3

$$\left(\frac{\kappa}{Q+2(Q-1)\kappa},\ 1-\frac{1-\lambda+q\kappa}{Q+2(Q-1)\kappa}\right)$$

is an exponent pair. Hence, by Theorem 2,

$$\left(\tfrac{1}{2}-\frac{1-\lambda+q\kappa}{Q+2(Q-1)\kappa},\ \tfrac{1}{2}+\frac{\kappa}{Q+2(Q-1)\kappa}\right),$$

i.e. (k, l) is an exponent pair, provided that

$$2k+l \geqslant 1.$$

This condition is equivalent to

$$1-2\,\frac{1-\lambda+q\kappa}{Q+2(Q-1)\kappa}+\tfrac{1}{2}+\frac{\kappa}{Q+2(Q-1)\kappa} \geqslant 1,$$

i.e. $$2\{2-2\lambda+(2q-1)\kappa\} \leqslant Q+2(Q-1)\kappa,$$

and this is true because

$$2(Q-1)\kappa \geqslant 2(2q-1)\kappa, \qquad \text{since } \kappa \geqslant 0,$$

and $$Q \geqslant 2 \geqslant 4(1-\lambda), \qquad \text{since } \lambda \geqslant \tfrac{1}{2}.$$

Therefore the condition is satisfied for all values of q and the theorem is completed.

4. THEOREM 5. $$\zeta(\tfrac{1}{2}+it) = O\left(t^{\frac{229}{1392}}\right).$$

We begin by proving that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{2}(l+k-\frac{1}{2})}), \tag{1}$$

where (k,l) is any exponent pair such that $l-k > \frac{1}{2}$. We have*

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} + \chi \sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}-it} + O(t^{-\frac{1}{4}}),$$

where $\chi = O(1)$.

The inequality (1) will follow from this if we can prove that

$$\sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} = O(t^{\frac{1}{2}(l+k-\frac{1}{2})}). \tag{2}$$

Now the function $f(n) = (t/2\pi)\log n$ satisfies the condition (3) in the definition of an exponent pair in § 3, if we take $y = t/2\pi$, $s = 1$, $u = 2$, $c = \frac{1}{3}$, and $r = 5$. And if further $1 \leqslant a < b < 2a$ and $z = t/2\pi a$ the conditions (2) are satisfied. Therefore, since (k,l) is an exponent pair, we have

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i(t/2\pi)\log n} = O(z^k a^l),$$

i.e.

$$\sum_{a\leqslant n\leqslant b} n^{it} = O\{(t/a)^k a^l\} = O(t^k a^{l-k})$$

for $1 \leqslant a < b < 2a < t/\pi$. Hence by partial summation we have

$$\sum_{a\leqslant n\leqslant b} n^{-\frac{1}{2}+it} = O(t^k a^{l-k-\frac{1}{2}}).$$

If t is large enough to ensure that $1 < \sqrt{(t/2\pi)} < t/\pi$, we can apply this with $a = 1$, $b = 1$; $a = 2$, $b = 3$; ... $a = 2^m$, $b = 2^{m+1}-1$;...; the last value of b being $[\sqrt{(t/2\pi)}]$; and then adding we have, since $l-k-\frac{1}{2} > 0$,

$$\sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} = O\{t^k t^{\frac{1}{2}(l-k-\frac{1}{2})}\} = O\{t^{\frac{1}{2}(l+k-\frac{1}{2})}\},$$

which proves (2), and therefore (1).

Now in § 3 we showed that $(0, 1)$ is an exponent pair. Hence, by Theorem 2, so is $(\frac{1}{2}, \frac{1}{2})$. Applying Theorem 4 with $q = 2$ we see that

$$\left(\frac{1}{2}-\frac{1-\frac{1}{2}+2.\frac{1}{2}}{4+6.\frac{1}{2}},\qquad \frac{1}{2}+\frac{\frac{1}{2}}{4+6.\frac{1}{2}}\right),$$

i.e. $(\frac{2}{7}, \frac{4}{7})$, is an exponent pair. Applying Theorem 4 with $q = 2$ to this last pair we see that

$$\left(\frac{1}{2}-\frac{1-\frac{4}{7}+2.\frac{2}{7}}{4+6.\frac{2}{7}},\qquad \frac{1}{2}+\frac{\frac{2}{7}}{4+6.\frac{2}{7}}\right),$$

i.e. $(\frac{13}{40}, \frac{22}{40})$, is an exponent pair. Again applying Theorem 4 with $q = 3$ we see that

$$\left(\frac{1}{2}-\frac{1-\frac{22}{40}+3.\frac{13}{40}}{8+14.\frac{13}{40}},\qquad \frac{1}{2}+\frac{\frac{13}{40}}{8+14.\frac{13}{40}}\right),$$

* See Titchmarsh (3), 32–4.

i.e. $(\frac{97}{251}, \frac{132}{251})$, is an exponent pair. Finally applying Theorem 1 to this last pair we see that

$$\left(\frac{\frac{97}{251}}{2(1+\frac{97}{251})}, \quad \tfrac{1}{2}+\frac{\frac{132}{251}}{2(1+\frac{97}{251})}\right),$$

i.e. $(\frac{97}{696}, \frac{480}{696})$, is an exponent pair. Since $\frac{480}{696}-\frac{97}{696} > \frac{1}{2}$, we can use this pair in (1), and we obtain

$$\zeta(\tfrac{1}{2}+it) = O\big(t^{\frac{229}{1392}}\big).$$

This completes the theorem.

4.1. THEOREM 6.

$$\zeta(\sigma+it) = O\left[t^{\frac{1}{4R-2}\left\{\frac{240Rr-16R+128}{240Rr-15R+128}\right\}}\right],$$

where $R = 2^{r-1}$ *on each of the lines* $\sigma = 1-\dfrac{r+1}{4R-2}$; $r = 3, 4, 5,\dots$.

We start from the approximate functional equation*

$$\zeta(\sigma+it) = \sum_{n<t/\pi} n^{-\sigma-it}+O(t^{-\sigma}) \qquad (t > 1). \tag{1}$$

As in the proof of Theorem 5 we have, for any exponent pair (k,l),

$$\sum_{a\leqslant n\leqslant b} n^{-it} = O(t^k a^{l-k}) \qquad (1 \leqslant a < b < 2a < t/\pi). \tag{2}$$

We now choose a set (k_q, l_q) of exponent pairs as follows: Applying Theorem 4 with $q = 4$ to the pair $(\frac{1}{2}, \frac{1}{2})$ we obtain the pair $(\frac{13}{31}, \frac{16}{31})$. Then applying Theorem 4 with $q = 1$ to this we obtain the pair $(\frac{16}{88}, \frac{57}{88})$. Finally, applying Theorem 3, we obtain the set of pairs

$$(k_q, l_q) = \left(\frac{16}{120Q-32}, \qquad 1-\frac{16q+31}{120Q-32}\right) \qquad (Q = 2^q).$$

If we put $x_q = l_q-k_q$ and $\sigma_q = 1-\dfrac{q+2}{4Q-2}$, then for every $q \geqslant 2$ we have $x_q > \sigma_q > x_{q-1}$. Using (k_q, l_q) in (2) we have, by partial summation,

$$\sum_{a\leqslant n\leqslant b} n^{-\sigma_q-it} = O(t^{k_q} a^{x_q-\sigma_q}), \tag{3}$$

and using (k_{q-1}, l_{q-1}) similarly we have

$$\sum_{a\leqslant n\leqslant b} n^{-\sigma_q-it} = O(t^{k_{q-1}} a^{-(\sigma_q-x_{q-1})}) \tag{4}$$

for $1 \leqslant a < b < 2a < t/\pi$.

We divide the sum on the right-hand side of (1) into two parts

* See, for example, Titchmarsh (3), Theorem 19. We have taken $x = t/\pi$, $c = \frac{3}{2}$.

according as $n < t^\lambda$ or $n \geqslant t^\lambda$ $(0 < \lambda < 1)$. For the first part we use (3) and for the second part we use (4). We thus have

$$\sum_{n<t^\lambda} n^{-\sigma_q - it} = \sum_{\frac{1}{2}t^\lambda \leqslant n < t^\lambda} + \sum_{\frac{1}{4}t^\lambda \leqslant n < \frac{1}{2}t^\lambda} + \dots$$
$$= O[t^{k_q+\lambda(x_q-\sigma_q)}\{(\tfrac{1}{2})^{x_q-\sigma_q} + (\tfrac{1}{4})^{x_q-\sigma_q} + \dots\}]$$
$$= O[t^{k_q+\lambda(x_q-\sigma_q)}], \qquad \text{since } x_q - \sigma_q > 0. \tag{5}$$

And

$$\sum_{t^\lambda \leqslant n < t/\pi} n^{-\sigma_q - it} = \sum_{t^\lambda \leqslant n < 2t^\lambda} + \sum_{2t^\lambda \leqslant n < 4t^\lambda} + \dots$$
$$= O[t^{k_{q-1}-\lambda(\sigma_q - x_{q-1})}\{1 + 2^{-(\sigma_q - x_{q-1})} + 4^{-(\sigma_q - x_{q-1})} + \dots\}]$$
$$= O[t^{k_{q-1}-\lambda(\sigma_q - x_{q-1})}], \qquad \text{since } \sigma_q - x_{q-1} > 0. \tag{6}$$

The right-hand sides of (5) and (6) are of the same order if

$$k_q + \lambda(x_q - \sigma_q) = k_{q-1} - \lambda(\sigma_q - x_{q-1}),$$

i.e. if

$$\lambda = \frac{k_{q-1} - k_q}{x_q - x_{q-1}}.$$

It is easily seen on substitution for k_q, etc., that this value of λ lies between 0 and 1. Hence, putting it in (5) and (6) and adding, we have

$$\sum_{n<t/\pi} n^{-\sigma_q - it} = O[t^{\mu_q}], \tag{7}$$

where

$$\mu_q = \frac{k_q(\sigma_q - x_{q-1}) + k_{q-1}(x_q - \sigma_q)}{x_q - x_{q-1}}.$$

Putting in the values of k_q, x_q, σ_q, etc., we obtain

$$\mu_q = \frac{1}{4Q-2}\left\{\frac{240Qq + 224Q + 128}{240Qq + 225Q + 128}\right\}.$$

We have proved this for each $q \geqslant 2$. If we put $q = r-1$, $R = 2^{r-1} = Q$, σ_q becomes $1 - \dfrac{r+1}{4R-2}$ and μ_q becomes

$$\mu_{r-1} = \frac{1}{4R-2}\left\{\frac{240Rr - 16R + 128}{240Rr - 15R + 128}\right\}.$$

Putting these in (7) and substituting in (1) we prove the theorem.

4.2. *Note to Theorems* 5 *and* 6. The results given in these two theorems are not the best that can be obtained. It seems very probable that no exponent pair arrived at by a finite number of applications of Theorems 1, 2, 3, and 4 will give a result which cannot be improved by further applications of the theorems. But the improvements are always extremely small while the results become very unwieldy. Unfortunately there appears to be no way of finding

the limit (we have found the exponent pair giving the best result after twelve applications of Theorem 4 and failed to disclose any recurrence in the values of q used in that theorem). We therefore content ourselves with giving results which are new and yet are not ridiculously clumsy.

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CONVEX REGIONS IN THE GEOMETRY OF PATHS—ADDENDUM

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1. In an affine space of paths there are, according to the paper referred to in the title,* ovaloid hypersurfaces. Any two points inside one of these can be joined by one, and only one, segment of a path which does not meet the boundary. In a footnote to p. 34 I stated that the proof applies only to paths defined by an affine connexion whose components are functions of position, as distinct from paths given by equations of the form

$$\frac{d^2x^i}{ds^2} = H^i\left(x, \frac{dx}{ds}\right), \tag{1.1}$$

the functions $H^i(x, \xi)$ being homogeneous of the second degree in ξ. This was an oversight, for by trivial modifications the theorem can be extended to paths given by equations of the form (1.1) with comparatively mild conditions imposed upon the functions H.

The main point of this generalization is that it covers the case of extremals obtained by varying an integral of the form

$$J = \int_{t_0}^{t_1} F(x, \dot{x})\, dt, \tag{1.2}$$

where $F(x, \xi)$ is positively homogeneous of the first degree in ξ, and

$$F_{\xi^i \xi^j} \lambda^j \neq 0 \quad \text{for} \quad \lambda^i \neq \rho \xi^i.$$

2. Assume H^i to be continuous for

$$|x^i| \leqslant 2, \qquad |\xi^i| \leqslant \lambda$$

and to satisfy a Lipschitz condition

$$|H^i(x_1, \xi_1) - H^i(x_0, \xi_0)| \leqslant \lambda^2 \alpha \sum |x_1^j - x_0^j| + \lambda\beta \sum |\xi_1^j - \xi_0^j|.$$

Then as far as the second and third sections of C.R. are concerned one or two minor changes of notation are sufficient for our purpose.

It only remains to construct a hypersurface $V(x) = 0$, such that

$$\overline{\overline{V}}(x, \xi) = \frac{\partial^2 V}{\partial x^j \partial x^k} \xi^j \xi^k + \frac{\partial V}{\partial x^i} H^i(x, \xi) \tag{2.1}$$

* *Quart. J. of Math.* (Oxford), 3 (1932), 33–42; referred to as C.R.

is positive at each point of (2.1) and for all non-zero vectors ξ. The hypersurfaces

$$V(x) \equiv x^i x^i - r^2 = 0 \tag{2.2}$$

satisfy this condition for all values of r less than some positive r_0. For there is a constant K such that

$$K > |H^i(0,\xi)|$$

if $\xi^i\xi^i = 1$. Hence there is a positive r_0 such that

$$K > |H^i(x,\xi)|$$

provided $\xi^i\xi^i = 1$ and $x^i x^i \leqslant r_0^2$. Evaluating (2.1) with V given by (2.2) and $r \leqslant r_0$, we have

$$\overline{\overline{V}}(x,\xi) = 2\xi^i\xi^i + 2x^i H^i(x,\xi).$$

But, for $\xi \neq 0$,

$$|x^i H^i(x,\xi)| < \sum_i |x^i| K\xi^p\xi^p \leqslant nKr\xi^p\xi^p,$$

and so the hypersurface (2.2) satisfies the required conditions if

$$r < 1/nK.$$

The remarks about Riemannian geometry on the last page of C.R. can be extended at once to spaces in which there is an integral of the form (1.2), or, in other words, a positive Finsler metric

$$ds^2 = g_{ij}(x,dx)\,dx^i dx^j,$$

where
$$g_{ij} = \frac{1}{2}\frac{\partial F^2}{\partial\, dx^i\, \partial\, dx^j}.$$

An existence proof for normal coordinates is to be found in § 9 of an article by O. Veblen and the present author.* Though it is tacitly assumed in this proof that the components Γ are functions of position, the argument applies just as well to paths defined by equations of the form (1.1). There is also a discussion of normal coordinates determined by an analytic field of extremals, in an article by E. Noether.† Generally speaking, if the functions H are of class k in x and ξ, the transformation to normal coordinates is of the same class, except possibly at the origin, where the second derivatives will be discontinuous unless

$$\Gamma^i_{jk} = -\frac{1}{2}\frac{\partial^2 H^i(x,\xi)}{\partial\xi^j\partial\xi^k}$$

are independent of ξ.

* *Proceedings of the Nat. Academy of Sciences*, 17 (1931), 551.

† *Göttinger Nachrichten*, 25 (1918), 37.

THE REPRESENTATION OF A NUMBER AS A SUM OF FIVE OR MORE SQUARES (II)

By E. MAITLAND WRIGHT (*Oxford*)

[Received 22 April 1933]

1. In this paper we improve certain results obtained in the former paper of the same title.* In fact, when $s = 5$, 6, or 7, we replace the previous value of α by a larger number and show that all our theorems are still true. The most noticeable improvement is in the case of five squares, where $\alpha = \frac{1}{11}$ is replaced by $\alpha = \frac{1}{8}$. When $s \geqslant 8$, the new method does not give any improvement.

We shall assume that the reader is familiar with the former paper. The notation is the same except that we now take $s = 5$, 6, or 7 and write

$$\alpha = \frac{s-4}{4(s-3)}, \qquad 0 < \beta < \alpha, \qquad \zeta = \frac{1}{2(s-3)}.$$

As before
$$\gamma = \tfrac{1}{2}s-1-(s-1)\beta, \qquad \delta = \alpha-\beta.$$

We now take ϵ any positive number such that $4s\epsilon < \delta$. The lemmas are numbered consecutively with those of the former paper; this enables us to quote our previous lemmas by number only. Lemmas 1–13 were proved under the hypothesis that

$$0 < \beta < 1, \qquad \epsilon > 0,$$

and so remain true under the new conditions.

2. Lemmas 18, 19, and 20 contain the new† idea which leads to the improvement.

Lemma 18. *If x lies on an arc $\mathfrak{M}$ for which $q > n^{\zeta}$, and if $0 < \beta < \alpha$, then*

$$|f_i(x)| < Cn^{\frac{1}{2}-\beta-\frac{1}{2}\zeta+2\epsilon}.$$

If x lies on $\mathfrak{M}'$, that is, if

$$|y| \leqslant q^{-\frac{2}{3}}n^{-\frac{2}{3}(1-\beta)},$$

we have, by Lemmas 10 and 11,

$$\begin{aligned} |f_i(x)| &\leqslant |f_i(x)-\psi_{\rho,i}(x)|+|\psi_{\rho,i}(x)| \\ &\leqslant C(q^{\frac{1}{2}+\epsilon}+q^{\frac{1}{2}+\epsilon}n^{1-\beta}|y|+n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}) \\ &\leqslant Cn^{\epsilon}(q^{\frac{1}{2}}+n^{\frac{1}{3}(1-\beta)}q^{-\frac{1}{6}}+n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}}). \end{aligned}$$

* *Quart. J. of Math.* (Oxford), 4 (1933), 37–51.

† That is, new as applied to this problem. The method is one used by Hardy and Littlewood in their work on Waring's problem for cubes and higher powers, but does not seem to have found any application in the case of squares.

If x lies on $\mathfrak{M}''$, that is, if

$$|y| > q^{-\frac{1}{3}}n^{-\frac{2}{3}(1-\beta)},$$

we have, by Lemma 13,

$$|f_i(x)| < Cn^{\epsilon}(q|y|)^{-\frac{1}{2}-\epsilon} < Cn^{2\epsilon+\frac{1}{3}(1-\beta)}q^{-\frac{1}{3}}.$$

In connexion with the arcs with which we are here concerned, we have $n^{\zeta} < q \leqslant n^{\frac{1}{2}(1-\beta)}$, and so

$$q^{\frac{1}{3}} \leqslant n^{\frac{1}{3}(1-\beta)}q^{-\frac{1}{3}} < n^{\frac{1}{3}(1-\beta)-\frac{1}{3}\zeta}, \qquad n^{\frac{1}{2}-\beta}q^{-\frac{1}{2}} < n^{\frac{1}{2}-\beta-\frac{1}{2}\zeta}.$$

Hence, on such an arc $\mathfrak{M}$,

$$|f_i(x)| < Cn^{2\epsilon}(n^{\frac{1}{3}(1-\beta)-\frac{1}{3}\zeta}+n^{\frac{1}{2}-\beta-\frac{1}{2}\zeta}).$$

Now
$$\tfrac{2}{3}\beta < \tfrac{2}{3}\alpha = \frac{s-4}{6(s-3)} = \tfrac{1}{6}-\tfrac{1}{3}\zeta,$$

and so
$$n^{\frac{1}{3}(1-\beta)-\frac{1}{3}\zeta} < n^{\frac{1}{2}-\beta-\frac{1}{2}\zeta}.$$

The lemma follows at once.

Lemma 19. *If* $0 < \beta < \frac{1}{2}$, *we have*

$$\int_{\mathfrak{K}} |f_1 f_2 f_3 f_4|\ |dx| < Cn^{1-2\beta+\epsilon}.$$

We have
$$f_1(x)f_2(x) = \sum_{j=N_1'+N_2'}^{N_1+N_2} R(j)x^j,$$

where $R(j)$ is the number of solutions of the equation

$$m_1^2+m_2^2 = j,$$

such that

$$m_1 > 0, \qquad m_2 > 0, \qquad N_1' \leqslant m_1^2 \leqslant N_1, \qquad N_2' \leqslant m_2^2 \leqslant N_2.$$

Then it is known* that

$$R(j) < Cj^{\epsilon} < Cn^{\epsilon}.$$

Also

$$\sum_{j=N_1'+N_2'}^{N_1+N_2} R(j) = f_1(1)f_2(1) < C(N_1^{\frac{1}{2}}-N_1'^{\frac{1}{2}})(N_2^{\frac{1}{2}}-N_2'^{\frac{1}{2}}) < Cn^{1-2\beta}.$$

Hence,
$$\int_{\mathfrak{K}} |f_1 f_2|^2\ |dx| = 2\pi \sum_{j=N_1'+N_2'}^{N_1+N_2} \{R(j)\}^2 < Cn^{1-2\beta+\epsilon}.$$

Similarly,
$$\int_{\mathfrak{K}} |f_3 f_4|^2\ |dx| = Cn^{1-2\beta+\epsilon},$$

and so

$$\left(\int_{\mathfrak{K}} |f_1 f_2 f_3 f_4|\ |dx|\right)^2 \leqslant \int_{\mathfrak{K}} |f_1 f_2|^2\ |dx| \int_{\mathfrak{K}} |f_3 f_4|^2\ |dx| < Cn^{2-4\beta+2\epsilon}.$$

* Landau, *Vorlesungen über Zahlentheorie*, i, Satz 262.

LEMMA 20. *If $s \geqslant 5$ and $0 < \beta < \alpha$, we have*

$$I_4' = \sum_{q > n^\zeta} \int_{\mathfrak{M}} |\prod f_i| \, |dx| < Cn^{\gamma - \frac{1}{2}\delta}.$$

By Lemmas 18 and 19,

$$I_4' < Cn^{(s-4)(\frac{1}{2}-\beta-\frac{1}{2}\zeta+2\epsilon)} \int_{\mathfrak{K}} |f_1 f_2 f_3 f_4| \, |dx|$$

$$< Cn^{(s-4)(\frac{1}{2}-\beta-\frac{1}{2}\zeta)+1-2\beta+2s\epsilon}.$$

Now

$$(s-4)(\tfrac{1}{2}-\beta-\tfrac{1}{2}\zeta)+1-2\beta = \tfrac{1}{2}s-1-(s-1)\beta-\tfrac{1}{2}\zeta(s-4)+\beta$$

$$= \gamma-\alpha+\beta = \gamma-\delta.$$

Since $2s\epsilon < \frac{1}{2}\delta$, the lemma is proved.

3. Lemmas 21–4 correspond to Lemmas 14–17 of the former paper. We write $\sum'$ to denote summation over those arcs for which $q \leqslant n^\zeta$.

LEMMA 21. *If $s = 5$, 6, or 7, $0 < \beta < \alpha$, and*

$$E_1' = n^{\beta-1} \sum_{\mathfrak{M}'}' q^{\frac{1}{2}s}, \qquad E_2' = n^{\frac{1}{2}(s-3)-\beta(s-2)} \sum_{\mathfrak{M}'}' q^{-\frac{1}{2}(s-2)},$$

$$E_3' = n^{\frac{1}{3}(s-2)(1-\beta)} \sum_{\mathfrak{M}'}' q^{-\frac{1}{6}(s+4)}, \qquad E_4' = n^{\frac{1}{2}(s-4)-\frac{2}{3}(s-1)\beta} \sum_{\mathfrak{M}'}' q^{\frac{1}{6}(s-4)},$$

then $$E_t' < Cn^{\gamma-\delta} \qquad (t = 1, 2, 3, 4).$$

Since $q \leqslant n^\zeta < n^{\frac{1}{3}(1-\beta)}$, we have

$$q^{\frac{2}{3}(s+1)} \leqslant n^{\frac{1}{3}(s+1)(1-\beta)},$$

$$n^{\beta-1} q^{\frac{1}{2}s} \leqslant n^{\frac{1}{3}(s-2)(1-\beta)} q^{-\frac{1}{6}(s+4)}.$$

Hence, every term of E_1' is less than or equal to the corresponding term of E_3', and so

$$E_1' \leqslant E_3'.$$

If ω is any real number greater than -2, depending only on s and β, then

$$\sum_{\mathfrak{M}'}' q^\omega \leqslant \sum_{q \leqslant n^\zeta} q^{\omega+1} < Cn^{\zeta(\omega+2)}.$$

Hence,

$$\sum_{\mathfrak{M}'}' q^{-\frac{1}{2}(s-2)} \leqslant \sum_{\mathfrak{M}'}' q^{-\frac{3}{2}} < Cn^{\frac{1}{2}\zeta}.$$

Then

$$E_2' = n^{\gamma-\frac{1}{2}+\beta} \sum_{\mathfrak{M}'}' q^{-\frac{1}{2}(s-2)} < Cn^{\gamma-\frac{1}{2}+\beta+\frac{1}{2}\zeta} < Cn^{\gamma-\delta},$$

since

$$\beta+\tfrac{1}{2}\zeta-\tfrac{1}{2} < \alpha+\tfrac{1}{2}\zeta-\tfrac{1}{2} = -\tfrac{1}{4} < -\alpha < -\delta.$$

Also,

$$E_3' < Cn^{\frac{1}{3}(s-2)(1-\beta)+\frac{1}{6}\zeta(8-s)},$$

$$E_4' < Cn^{\frac{1}{2}(s-4)-\frac{2}{3}(s-1)\beta+\frac{1}{6}\zeta(s+8)}.$$

Then it only remains to show that

$$\tfrac{1}{3}(s-2)(1-\beta)+\tfrac{1}{6}\zeta(8-s) \leqslant \gamma-\delta,$$
$$\tfrac{1}{6}(s-4)-\tfrac{2}{3}(s-1)\beta+\tfrac{1}{6}\zeta(s+8) \leqslant \gamma-\delta.$$

Putting $\delta = \alpha-\beta$ and rearranging, we see that these reduce to

$$\tfrac{2}{3}(s-2)\beta \leqslant \tfrac{2}{3}(s-2)\alpha,$$
$$\frac{s-4}{3}\beta \leqslant \frac{4(s-1)}{3}\alpha,$$

both of which are clearly satisfied. Lemma 21 is therefore true.

We write

$$I_1' = \sum_{\mathfrak{M}'}{}' \int_{\mathfrak{M}'} |\prod f_i(x)-\prod \psi_{p,i}(x)|\ |dx|,$$

$$I_2' = \sum_{\mathfrak{M}''}{}' \int_{\mathfrak{M}''} |\prod f_i(x)|\ |dx|,$$

$$I_3' = \sum_{\mathfrak{M}'}{}' \int_{\mathfrak{K}-\mathfrak{M}'} |\prod \psi_{p,i}(x)|\ |dx|.$$

LEMMA 22. *If $s = 5$, 6, or 7, and $0 < \beta < \alpha$, then*

$$I_1' < Cn^{s\epsilon}(E_1'+E_2'+E_3') < Cn^{\gamma-\frac{1}{2}\delta},$$
$$I_2' < Cn^{2s\epsilon}E_3' < Cn^{\gamma-\frac{1}{2}\delta},$$
$$I_3' < Cn^{s\epsilon}E_4' < Cn^{\gamma-\frac{1}{2}\delta}.$$

This lemma may be proved in the same way as Lemmas 15, 16, and 17.

4. THEOREM 1A. *If $s = 5$, 6, or 7, and if $0 < \beta < \alpha$, then*

$$r(n) = \frac{T}{2^s\Gamma(s)(\prod \lambda_i)^{\frac{1}{2}}}\mathfrak{S}(n)n^{\gamma}+O(n^{\gamma-c}),$$

where $c = c(s,\beta) > 0$.

As in the proof of Theorem 1, we may show that the coefficient of x^n in

$$\sum_{q\leqslant n^{\zeta}}\sum_{p} \{\prod \psi_{p,i}(x)\}$$

is

$$\mu(n)\mathfrak{S}(n, n^{\zeta}).$$

Then

$$r(n)-\mu(n)\mathfrak{S}(n,n^{\zeta})$$
$$= \frac{1}{2\pi i}\int_{\mathfrak{K}} \Big\{\prod f_i(x)-\sum_{q\leqslant n^{\zeta}}\sum_{p}\prod \psi_{p,i}(x)\Big\}\frac{dx}{x^{n+1}}$$

$$= \frac{1}{2\pi i}\left\{\sum_{\mathfrak{M}'}{}' \int_{\mathfrak{M}'} \left(\prod f_i - \prod \psi_{\rho,i}\right) \frac{dx}{x^{n+1}} + \sum_{\mathfrak{M}''}{}' \int_{\mathfrak{M}''} \left(\prod f_i\right) \frac{dx}{x^{n+1}} - \right.$$

$$\left. - \sum_{\mathfrak{M}'}{}' \int_{\mathfrak{K}-\mathfrak{M}'} \left(\prod \psi_{\rho,i}\right) \frac{dx}{x^{n+1}} + \sum_{q>n^{\zeta}} \int_{\mathfrak{M}} \left(\prod f_i\right) \frac{dx}{x^{n+1}}\right\}.$$

Hence, by Lemmas 20 and 22,

$$|r(n) - \mu(n)\mathfrak{S}(n, n^{\zeta})| \leqslant \frac{1}{2\pi}(I_1' + I_2' + I_3' + I_4') < Cn^{\gamma - \frac{1}{2}\delta}.$$

By the method of Lemma 7 we can show that

$$|\mathfrak{S}(n) - \mathfrak{S}(n, n^{\zeta})| \leqslant Cn^{-\frac{1}{2}\zeta}.$$

The rest of the proof is the same as that of Theorem 1.

Theorems 2 A and 3 A, corresponding to Theorems 2 and 3, follow as before.

RELATIVE MEASURABILITY AND THE DERIVATES OF NON-MEASURABLE FUNCTIONS

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Introduction

WE have recently given proofs of two theorems which embody the facts known about the possible dispositions of the derivates of measurable functions.† Our work originated in the study of a memoir of Besicovitch's and we mentioned that a discussion on similar lines to his had been given by Saks.‡ The theorems of our paper went further than those of Saks in that they gave information about the approximate derivates as well as about the derivates of a measurable function. In another respect, however, Saks's results had an advantage over ours, since he did not assume his functions to be measurable.

It was natural, then, that we should extend our definitions of approximate derivates to apply to non-measurable functions and set out to obtain theorems generalizing our earlier ones so as to include those of Saks.

We formed a convenient basis for this extension by developing the concept of relative measurability, which had been mentioned by Hausdorff.§ In Theorems 1–3 we summarize the simplest properties of relatively measurable sets, some of which had been given by Hausdorff; the proofs are easy and we omit them. Theorems 4–8 give further properties of relatively measurable sets and functions. We establish in Theorem 9 the extension to approximate derivates of a theorem of Banach's dealing with ordinary derivates;|| our result is that a function having one of its approximate derivates finite in

† *Proc. London Math. Soc.* (2) 32 (1931), 346–55.

‡ A. S. Besicovitch, *Bull. de l'Académie des Sciences de Russie* (1925), 97–122 and 527–40; S. Saks, *Fundamenta Math.* 5 (1924), 98–104.

§ *Grundzüge der Mengenlehre* (1914), 415. See also Pierpont, *Theory of Functions of Real Variables*, 2 (1912), 366, and Hildebrandt, *Bull. American Math. Soc.* 24 (1918), 125.

|| S. Banach, *Fundamenta Math.* 3 (1922), 128–32, Theorem II.

a set is measurable in relation to that set. This enables us to deduce differential properties of a general function from those of an associated measurable function.

Our work was still incomplete when Mr. A. J. Ward showed us some work of his on general functions. His analysis had points of contact with ours, but his methods appeared to have some advantages. It seemed appropriate, therefore, to leave to Mr. Ward any further developments and the final statement of the cases which may occur in the structure of a general function, a statement which will extend that of § 7 of our paper on measurable functions.† We are indebted to Mr. Ward for completing the enunciation and the proof of Theorem 10; as it is set out in this paper, it contains the application of his arguments to the 'associated measurable function' which we had constructed.

Relative measurability

We define the statement that a set E_1 *is measurable in relation to a* set E_0, which we shall write as $E_1 \mu E_0$.

DEFINITION. *$E_1 \mu E_0$ if there is a measurable set M such that $E_0 E_1 = E_0 M$.*

On this definition we make two remarks. The first is that we may assume M to be such that $mM = m^*(E_0 E_1)$. For if M_1 is any measurable set such that $E_0 E_1 = E_0 M_1$ and M_2 is a measurable set containing $E_0 E_1$ such that $mM_2 = m^*(E_0 E_1)$, then $M = M_1 M_2$ has the required properties. Secondly, the question whether E_1 is measurable in relation to E_0 depends only on the sub-set of E_1 contained in E_0; hence we shall not lose generality in supposing in the future that E_1 is contained in E_0.

THEOREM 1. (*a*) *If $E_1 \mu E_0$, then $(E_0 - E_1) \mu E_0$.*

(*b*) *If $E_n \mu E_0$ for $n = 1, 2, \ldots$, then*

$$(E_1 + E_2 + \ldots) \mu E_0 \quad \textit{and} \quad (E_1 E_2 \ldots) \mu E_0.$$

(*c*) *If $E_n \mu E_0$ for $n = 1, 2, \ldots$, then $(\overline{\lim} E_n) \mu E_0$ and $(\underline{\lim} E_n) \mu E_0$.*

† We take this opportunity of saying that the last remark of that paragraph, that, in two particular respects, Besicovitch had proved more than Denjoy, was unjustified. Although Besicovitch established the two results in question in a more direct way, they had been obtained by Denjoy in the course of his researches into integration and were mentioned in a footnote in his paper, 'Mémoire sur la totalisation des nombres dérivés non-sommables': *Ann. de l'École Normale* (3) 33 (1916), 209.

(*d*) *If* $E_1 \mu E_0$ *and* $E_2 \mu E_0$, *where* $E_2 \subset E_1 \subset E_0$, *then* $E_2 \mu E_1$.

(*e*) *If* M_0, M_1 *are measurable and* E *is any set, then* $(M_1 E) \mu (M_0 E)$.

THEOREM 2. *A necessary and sufficient condition that* $E_1 \mu E_0$ *is that*

$$m^*E_1 + m^*(E_0 - E_1) = m^*E_0$$

(*where* m^*E_0 *is supposed finite*).

THEOREM 3. *If* E_1, E_2 *are sets contained in* E_0, *having no common points, such that* $E_1 \mu E_0$, $E_2 \mu E_0$, *then*

$$m^*E_1 + m^*E_2 = m^*(E_1 + E_2).$$

COROLLARY. *The theorem extends to an enumerable infinity of sets* E_n.

THEOREM 4. *A necessary and sufficient condition that* $E_1 \mu E_0$ *is that* $E_0 - E_1$ *has outer density zero at almost all points of* E_1.

We prove the necessity. With the exception of a set of measure zero, we have, if x is a point of E_1 and I_n a sequence of intervals with x as an end-point whose lengths tend to 0,

$$\lim \frac{m^*(E_1 I_n)}{mI_n} = 1.$$

Since $E_1 \mu E_0$, it is clear that $(E_1 I_n) \mu (E_0 I_n)$ and so from Theorem 2

$$m^*E_1 I_n + m^*(E_0 - E_1)I_n = m^*E_0 I_n.$$

Therefore
$$\lim \frac{m^*(E_0 - E_1)I_n}{mI_n} = 0.$$

We now prove the sufficiency. Let M_1, M_2 be measurable sets containing respectively the sets E_1, $E_0 - E_1$ and such that

$$mM_1 = m^*E_1, \qquad mM_2 = m^*(E_0 - E_1).$$

It is sufficient to prove that $m(M_1 M_2) = 0$. If not, then at almost every point of $M_1 M_2$, the set M_2 has density 1. This shows that, at a set of points x of E_1 of positive outer measure,

$$\lim \frac{m^*(E_0 - E_1)I_n}{mI_n} = 1,$$

where I_n is a decreasing sequence of intervals with limit-point x, and this contradicts the condition of the theorem.

It may be remarked that the sufficiency argument holds if we assume only that one of the lower outer densities of $E_0 - E_1$ is less than 1 almost everywhere in E_1. We then have a sufficient condition which is wider than the necessary condition and the gap between the two conditions embodies the following result:

If E is any set, then at almost all points outside E the outer density of E exists and is equal to either 0 *or* 1.

It is well known that at almost all points of E the outer density of E is 1. We may therefore sum up in the following theorem.

THEOREM 5. *If E is any set, then, at almost all points, the outer density of E is defined and is equal to* 0 *or* 1.

It is convenient to give here a theorem closely connected with Theorem 5, which will be required for the proof of Theorem 7. Let HE denote the set E together with the set of points at which the outer density of E is 1.

THEOREM 6. *The set HE is measurable.*

Let M be a measurable set containing E, with $mM = m^*E$.

Then, if I is any interval, $m(MI) = m^*(EI)$, and it follows that at any point the upper and lower outer densities of E are respectively equal to the upper and lower densities of M. But the points of unit density of M form a measurable set (of measure mM), so that HE is measurable [and $m(HE) = m^*E$].

Relatively measurable functions

A function f is said to be measurable in relation to a set E (in symbols, $f \mu E$) if, for every A, the set for which $f \geqslant A$ is measurable in relation to E.

It is clear from Theorem 1 (*a*) and (*b*) that if one of the four sets

$$E(f \geqslant A), \qquad E(f > A), \qquad E(f \leqslant A), \qquad E(f < A)$$

is, for every A, measurable in relation to E, so are all the four.

THEOREM 7. *If $f \mu E$, then there is a measurable set M containing E and a measurable function ϕ defined in M, such that $\phi = f$ at all points of E.*

Proof. First suppose $f(x)$ bounded and assume for simplicity that $0 \leqslant f(x) < 1$.

Write $E(a, b)$ for the set in which $a \leqslant f(x) < b$.

Define $\phi_1(x)$ equal to

$$\begin{aligned} &f(x) \text{ for } x \text{ in } E, \\ &\tfrac{1}{2} \text{ for } x \text{ in } HE(\tfrac{1}{2}, 1) \text{ and not in } E, \\ &0 \text{ for } x \text{ in } HE(0, 1) \text{ and not in } HE(\tfrac{1}{2}, 1) + E \end{aligned}$$

(where, as in Theorem 6, the prefix H to any set means the addition of points at which its outer density is 1).

Define $\phi_n(x)$ equal to

$$f(x) \text{ for } x \text{ in } E,$$

$$\frac{2^n-1}{2^n} \text{ for } x \text{ in } HE\left(\frac{2^n-1}{2^n}, 1\right) \text{ and not in } E,$$

and, generally for $r = 2^n-1,..., 1, 0$, equal to

$$\frac{r}{2^n} \text{ for } x \text{ in } HE\left(\frac{r}{2^n}, 1\right) \text{ and not in } HE\left(\frac{r+1}{2^n}, 1\right)+E.$$

The set $e(r,n)$ in which $\phi_n(x) \geqslant r2^{-n}$ is $HE(r/2^n, 1)-E(0, r/2^n)$. Since $f \, \mu \, E$, we have $E(0, r/2^n) \, \mu \, E$ and, by Theorem 4, the set $E(r/2^n, 1)$ has outer density zero at almost all points of $E(0, r/2^n)$. Thus $e(r,n)$ differs from $HE(r/2^n, 1)$ by a set of measure zero and so by Theorem 6 it is measurable.

Clearly $\phi_n(x)$ increases with n for every x in HE. If $\phi(x)$ is the limit function of the sequence, $\phi(x) = f(x)$ in E. It remains to prove that $\phi(x)$ is measurable.

Let E_A denote the set in which $\phi(x) \geqslant A$ and let A be the limit of an increasing sequence

$$\frac{r_1}{2^{n_1}} < ... < \frac{r_k}{2^{n_k}} < ... < A.$$

Then
$$E_A = \prod_{k=1}^{\infty} e(r_k, n_k)$$

and so E_A is measurable. Hence ϕ is a measurable function.

There is no difficulty in making the extension to an unbounded $f(x)$ by considering separately the sets for which

$$r \leqslant f(x) < r+1 \qquad (r = ...-1, 0, 1,...).$$

It is to be observed in this theorem that if we take M such that $mM = m^*E$, then ϕ is unique, if we disregard difference in a set of measure zero. For two functions ϕ_1, ϕ_2 can differ only in a measurable sub-set of $M-E$.

We state now a converse theorem, which is easily proved by use of Theorem 1 (e).

Theorem 8. *If $\phi(x)$ is measurable in a measurable set M and if E is any sub-set of M, then $\phi_E(x) \, \mu \, E$.*

Here $\phi_E(x)$ denotes the function $\phi(x)$ considered only at points of E.

Approximate derivates of non-measurable functions

Let $AD^+(f,x,\lambda)$ be the lower bound of numbers a such that the set of points ξ for which

$$f(\xi)-f(x) \geqslant a(\xi-x) \qquad (\xi > x)$$

has upper right-hand outer density at x less than or equal to λ.

As λ decreases, $AD^+(f,x,\lambda)$ increases and we define the upper right approximate derivate $AD^+f(x)$ to be

$$\lim_{\lambda\to 0} AD^+(f,x,\lambda).$$

For clearness, we add the definition of lower right derivates.

We define $AD_+(f,x,\lambda)$ as the upper bound of numbers a such that the set of ξ for which

$$f(\xi)-f(x) \leqslant a(\xi-x)$$

has upper right-hand outer density less than or equal to λ.

Then $AD_+(f,x,\lambda)$ decreases as λ decreases and we write $AD_+f(x)$ for

$$\lim_{\lambda\to 0} AD_+(f,x,\lambda).$$

We may notice the inequality

$$AD_+(f,x,1-\lambda-\epsilon) \leqslant AD^+(f,x,\lambda).$$

The definitions of left derivates will now be clear.

THEOREM 9. *If $\lambda = \lambda(x) < 1$ and $AD^+(f,x,\lambda) < +\infty$ in a sub-set E of the set E_0 in which f is defined, then $f\ \mu\ E$.*

If not, there is a value of A for which $E(f < A)$ is not measurable in relation to E.

By Theorem 4 there is, then, a sub-set E_1 of points x of $E(f < A)$ with $m^*E_1 > 0$ such that the set $E(f \geqslant A)$ has outer density 1 at x.

It is clear that at x, $AD^+(f,x,\lambda)$ taken over E (and, *a fortiori*, over E_0) is $+\infty$, which contradicts the hypothesis.

THEOREM 10. *Suppose $f(x)$ defined in E_0. For x in a set E, suppose that, for some λ less than 1, $AD^+(f,x,\lambda) < +\infty$. Then, at almost all points of E, for all λ, μ,*

$$AD^+(f,x,\lambda) = AD_-(f,x,\mu) = AD\phi(x)$$

(where ϕ is the measurable function, associated with f and E, defined in Theorem 7).

By Theorem 9, $f(x)\ \mu\ E$.

By Theorem 7, there exists a function $\phi(x)$ measurable in M (where $m^*E = mM$) and equal to $f(x)$ in E.

At points of E of outer density 1,

$$AD^{+}(\phi, x, \lambda) = AD^{+}(f_E, x, \lambda) \leqslant AD^{+}(f, x, \lambda).$$

Theorem 1 of our paper dealing with measurable functions shows that $AD\phi(x)$ exists at almost all points of M.

Suppose that in a sub-set E_1 of E of positive outer measure, for some λ,

$$AD^{+}(f, x, \lambda) > AD\phi(x).$$

It follows that, for x in E_1, the set of points x' of $E_0 M$ at which $f(x') > \phi(x')$ has upper outer density at x different from zero.

Let I be an interval containing a part E_2 of E_1 such that

$$m^*E_2 > \tfrac{3}{4}mI.$$

From Theorems 5 and 6, the set at which the upper outer density of any set is greater than 0 is measurable and has measure equal to the outer measure of that set.

Hence the set of points of I where $f(x) > \phi(x)$ has outer measure at least m^*E_2, therefore greater than $\tfrac{3}{4}mI$.

Choose ϵ such that the set E_3 of points of I where $f(x) > \phi(x)+\epsilon$ has outer measure greater than $\tfrac{1}{2}mI$.

The set of points of I at which E_3 has outer density 1 is measurable and has measure greater than $\tfrac{1}{2}mI$.

Therefore there is a point of E_2 at which E_3 has outer density 1. At such a point, clearly $AD^{+}(f, x, \lambda) = \infty$ for all λ, which is a contradiction.

An exactly similar argument shows that the hypothesis that $AD_{-}(f, x, \mu) < AD\phi(x)$ in a set of positive outer measure is untenable.

ON CONFORMAL REPRESENTATION—ADDENDUM

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IN a previous paper I obtained formulae giving the conformal representations upon the half-planes of variables ζ, Z' of curvilinear triangles of angles $\frac{1}{3}\pi$, 0, 0; $\frac{1}{6}\pi$, $\frac{1}{6}\pi$, 0 in the plane of the variable τ. In these formulae Z' is expressed as a two-valued function of ζ, but the formula expressing ζ as a function of τ is so complicated that I was unable to show explicitly that Z' is a one-valued function of τ, as the theory demands.* This defect is now removed by the employment of known results.

In the discussion of the formula for ζ I failed to observe that, after the substitution $\zeta = \xi^3$, my subsidiary equation

$$64z = \zeta(\zeta+8)^3(\zeta-1)^{-3}$$

assumes one of the forms of the tetrahedral equation,† which is solved by Klein in the form‡

$$\xi = \frac{2\sum\limits_{m=-\infty}^{\infty}(-1)^m(6m+1)q^{\frac{1}{12}(6m+1)^2}}{3\sum\limits_{m=-\infty}^{\infty}(-1)^m(2m+1)q^{\frac{3}{4}(2m+1)^2}} \qquad [q = \exp(i\pi\tau)].$$

The lengthy formula for ζ given in my paper may be replaced by the more compact $\zeta = \xi^3$.

The apparently two-valued factor $(\zeta-1)^{\frac{1}{2}}$ occurs in my formula for Z'. But ξ, $(\xi^3-1)^{\frac{1}{2}}$ are the automorphic functions associated with a certain congruence-group, and the following formulae are given:§

$$(\xi^3-1)^{\frac{1}{2}} = \frac{y(y^2-9)}{3\sqrt{3}(y^2-1)}, \qquad y = \frac{1}{q}\left\{\sum_{m=0}^{\infty} q^{m(m+1)}\Big/\sum_{m=0}^{\infty} q^{3m(m+1)}\right\}^2.$$

These formulae suffice to show that Z' is, in fact, a one-valued function of τ.

* J. Hodgkinson, *Quart. J. of Math.* (Oxford), 2 (1931), 20–30 (25–9).

† Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, i (1890), 104.

‡ K.-F. ii (1892), 375.

§ K.-F. i. 677, 689; ii. 391.